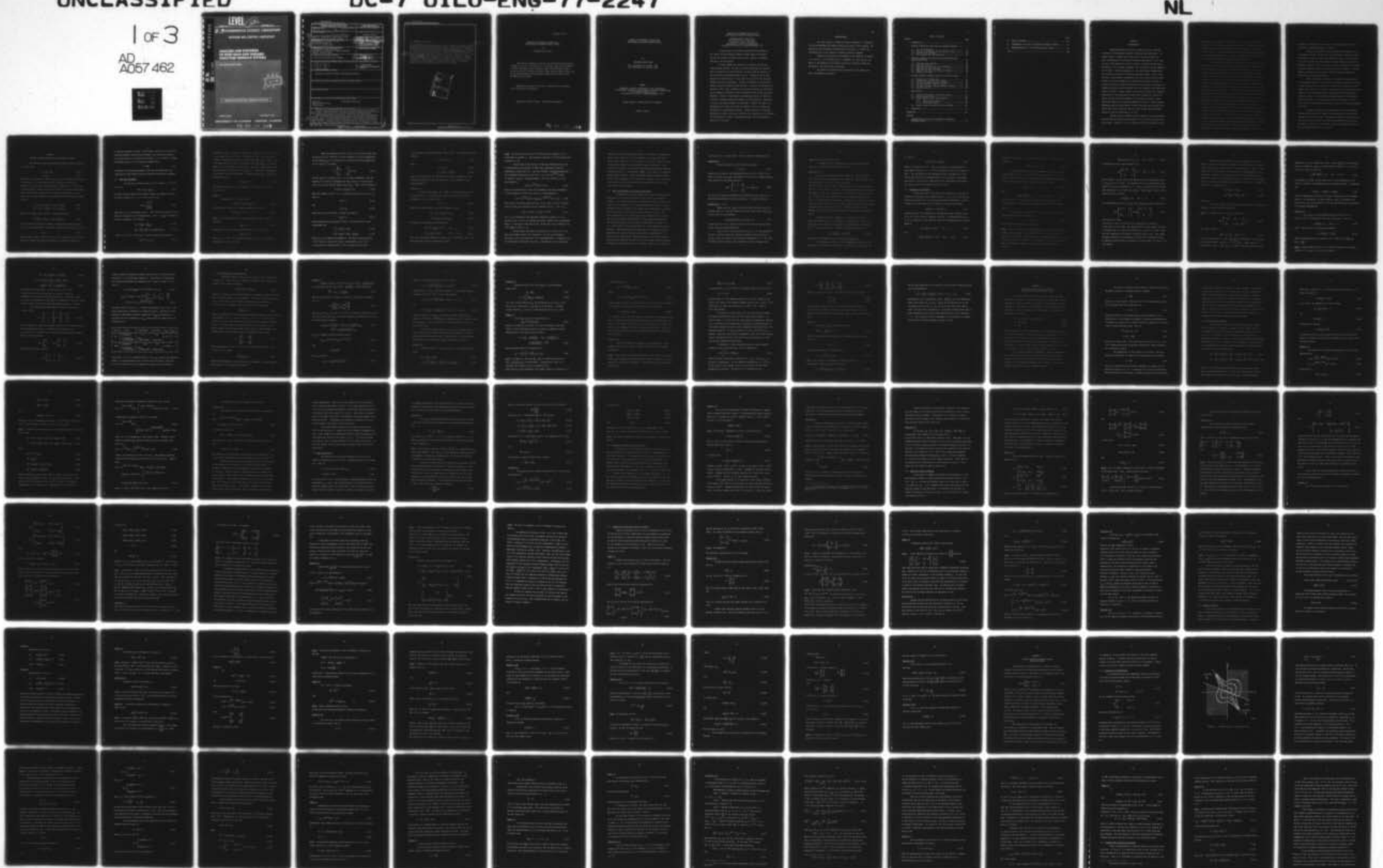
AD-A057 462 ILLINOIS UNIV AT URBANA-CHAMPAIGN DECISION AND CONTROL LAB F/G 9/3
ANALYSIS AND SYNTHESIS OF HIGH GAIN AND VARIABLE STRUCTURE FEED--ETC(U)
NOV 77 K D YOUNG DAAB07-72-C-0259
UNCLASSIFIED DC-7 UILU-ENG-77-2247 NL

1 OF 3
AD
A057 462



LEVEL II 12

REPORT DC-7 NOVEMBER, 1977

CSL COORDINATED SCIENCE LABORATORY

DECISION AND CONTROL LABORATORY

ANALYSIS AND SYNTHESIS OF HIGH GAIN AND VARIABLE STRUCTURE FEEDBACK SYSTEMS

KAR-KEUNG DAVID YOUNG

REPORT R-800 UILU-ENG 77-2247

UNIVERSITY OF ILLINOIS - URBANA, ILLINOIS

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. ~~REPORT NUMBER~~ (9) Doctoral thesis,

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)

(6) ANALYSIS AND SYNTHESIS OF HIGH GAIN AND VARIABLE STRUCTURE FEEDBACK SYSTEMS.

5. TYPE OF REPORT & PERIOD COVERED

Technical Report,

6. ~~PERFORMING~~ ORG. REPORT ~~NUMBER~~

(14) R-800 DC-73 UILU-ENG-77-2247

7. AUTHOR(s)

(10) Kar-Keung David Young ↖pg 181

8. ~~CONTRACT OR GRANT NUMBER(s)~~

(15) DAAB-07-72-C-0259, ~~NSF ENG-74-20091~~, AFOSR-73-2570

US ERDA EX-76-C-01-2088

9. PERFORMING ORGANIZATION NAME AND ADDRESS

Coordinated Science Laboratory
University of Illinois at Urbana-Champaign
Urbana, Illinois 61801

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS

Joint Services Electronics Program

12. ~~REPORT DATE~~

(11) Nove[redacted] 1977

13. ~~NUMBER OF PAGES~~

188

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

(12) 197p.

15. SECURITY CLASS. (of this report)

UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

High Gain
Singular Perturbation
Sliding Modes
Disturbance Rejection

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this study, we deal with the analysis and synthesis of two classes of multivariable feedback systems, high gain feedback systems and variable structure systems (VSS), subject to parameter variations and disturbances.

We first examine the insensitivity property of high gain state feedback systems. The idea of incorporating high gain feedback loops in observers is then introduced. We then study the behavior of the resulting "two-high-gain loops" system. A design procedure which enhances the insensitivity of the closed loop system for the high gain feedback matrices is developed. Second,

is studied

ory

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

410 775

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

1B

20. ABSTRACT (continued)

we establish the relationship between high gain feedback systems and VSS. is established We reveal the insensitivity property of VSS. Then, observers with variable structure are developed and we examine the behavior of the resulting VSS with variable structure observers. Third, ~~we investigate~~ the robustness property of these two classes of feedback systems with respect to model reduction is investigated. Neglection of acuator and sensor dynamics is considered. Finally, ~~we exploit~~ the insensitivity of these two classes of feedback control is used to enhance the reliability of interconnected systems. Design procedures of high gain feedback and variable structure controllers are illustrated in the designs for a distillation column, a synchronous machine and the longitudinal motions of an aircraft.

UILU-ENG 77-2247

ANALYSIS AND SYNTHESIS OF HIGH GAIN AND VARIABLE STRUCTURE FEEDBACK SYSTEMS

by

Kar-Keung David Young

This work was supported in part by the Joint Services Electronics Program (U.S. Army, U.S. Navy and U.S. Air Force) under Contract DAAB-07-72-C-0259; National Science Foundation under Grant NSF ENG 74-20091; Air Force Office of Scientific Research under Contract AFOSR 73-2570; and by the Energy Research and Development Administration under Contract US ERDA EX-76-C-01-2088.

ANALYSIS AND SYNTHESIS OF HIGH GAIN AND VARIABLE STRUCTURE FEEDBACK SYSTEMS

BY

KAR-KEUNG DAVID YOUNG

B.S., University of Illinois, 1974
M.S., University of Illinois, 1975

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Electrical Engineering
in the Graduate College of the
University of Illinois at Urbana-Champaign, 1978

Thesis Adviser: Professor Petar V. Kokotovic

Urbana, Illinois

ANALYSIS AND SYNTHESIS OF HIGH GAIN
AND VARIABLE STRUCTURE FEEDBACK SYSTEMS

Kar-Keung David Young, Ph.D.
Decision and Control Laboratory
Coordinated Science Laboratory
Department of Electrical Engineering
University of Illinois at Urbana-Champaign, 1978

In this study, we deal with the analysis and synthesis of two classes of multivariable feedback systems, high gain feedback systems and variable structure systems (VSS), subject to parameter variations and disturbances.

We first examine the insensitivity property of high gain state feedback systems. The idea of incorporating high gain feedback loops in observers is then introduced. We then study the behavior of the resulting "two-high-gain-loops" system. A design procedure which enhances the insensitivity of the closed loop system for the high gain feedback matrices is developed. Second, we establish the relationship between high gain feedback systems and VSS. We reveal the insensitivity property of VSS. Then, observers with variable structure are developed and we examine the behavior of the resulting VSS with variable structure observers. Third, we investigate the robustness property of these two classes of feedback systems with respect to model reduction. Neglection of acuator and sensor dynamics is considered. Finally, we exploit the insensitivity of these two classes of feedback control to enhance the reliability of interconnected systems. Design procedures of high gain feedback and variable structure controllers are illustrated in the designs for a distillation column, a synchronous machine and the longitudinal motions of an aircraft.

ACKNOWLEDGMENT

The author wishes to thank his advisor Professor P. V. Kokotovic, for his encouragement and guidance during the course of this research. He would also like to express his gratitude to Professor V. I. Utkin, for introducing him to the concepts of variable structure feedback.

Thanks are also given to the members of his committee, Professors J. B. Cruz, Jr., W. R. Perkins and J. V. Mendanic for their helpful comments, to the members of the Decision and Control Laboratory, both past and present, for many helpful discussions.

The author also expresses his appreciation to his family for their encouragment and support.

TABLE OF CONTENTS

Page

CHAPTER 1

INTRODUCTION

High gain feedback has been a classical tool for reduction of effects of disturbances, parameter variations and distortions. Although limited to single input-single output feedback systems, the early investigations of structures permitting high gains [1], the rules for root locus asymtotes [2] and the results on sensitivity and return difference [3,4] has greatly deepened the intuition of control engineers in the 1950's. Recent developments in the multivariable system theory have revived the interest in high gain feedback systems. First, recent studies on multivariable transmission zeros [5,6] and the work on multivariable root locus asymptotes [7] generalize the notion of zeros of transfer functions in single input-single output systems and consider the asymptotic behavior of multivariable high gain feedback systems when gain tends to infinity. Second, works on disturbance rejection [8], parameter uncertainty [9] and decoupling of large scale systems [10] either purposely introduce high gain feedback in the problem statement or they implicitly appear in the resulting feedback structures. Third, feedback implementations of linear optimal controls when only small penalties are made on the control variables (the so-called "cheap" control problem) result in loops with high gain [11,12,13].

Another class of feedback controls capable of reducing parameter sensitivities and rejecting disturbances is the so-called variable structure control. Basically, it is a feedback control discontinuous on some

switching surface defined on the state space. This class of feedback systems is called variable structure systems (VSS). It has been developed in the USSR in the last fifteen years [14,15,16,17,18]. For a recent survey, see [19]. A salient feature of VSS is the so-called sliding mode occurs on the switching surface. It is in sliding mode that VSS becomes less sensitive to parameter variations and disturbances.

In this study, we deal with the analysis and synthesis of these two classes of multivariable feedback systems, high gain feedback systems and VSS, subject to parameter variations and disturbances. First, we examine the insensitivity property of high gain state feedback systems. Then, in the case when only some of the states variables are accessible, we introduce the idea of incorporating high gain feedback loops in observers. By allowing high gain feedback in the observer structure, the observation error dynamics enjoy the same insensitivity property inherent in high gain feedback systems. In place of high gain state feedback, the "high gain" observer states are fed back through the main high gain feedback loops. We study the behavior of this "two-high-gain-loops" feedback system. A practical alternative to examining the sensitivity of all the states of high gain feedback systems is to investigate only the sensitivity of the variables which are critical to performance degradation or have to meet certain design specifications. We call these variables, the regulated variables. We discover that there remain some degrees of freedom in the design of the high gain feedback matrices which can be utilized to enhance the insensitivity of high gain feedback systems. The number of degrees of freedom depends on the number of control variables, available measurements, disturbance inputs and regulated variables. A

procedure is developed for the design of the feedback matrices which exercises the available degrees of freedom.

Second, we establish the relationship between high gain feedback systems and VSS. We find that in sliding mode, VSS enjoy the same insensitivity property in high gain feedback systems. This encourages us to develop observers with variable structure feedback in parallel to high gain observers. We then proceed to examine the behavior of variable structure feedback systems with variable structure observers. The same degrees of freedom in the design of variable structure feedback is discovered as in high gain systems.

Third, we address an important consideration in application: the robustness property of these classes of feedback systems with respect to model reduction. As a step in this direction, we consider the most common model reduction in practice, the neglection of actuator and sensor dynamics. This robustness property differentiates VSS and high gain feedback systems. As a reward for the added complexity, VSS are more robust with respect to the neglected small time constants. This is due to the fact that variable structure control does not force the motion to be fast.

Finally, we exploit the insensitivity property in high gain feedback systems and VSS to enhance the reliability of interconnected systems in the spirit of [20]. The design procedures of high gain feedback and variable structure controllers are illustrated by the design of control systems for a distillation column, a synchronous machine and the longitudinal motions of an aircraft.

CHAPTER 2

SINGULAR PERTURBATION AND HIGH GAIN FEEDBACK SYSTEMS

The high gain linear time-invariant feedback system we consider is of the form

$$\dot{x} = A_o x + B_o u \tag{2.1}$$

$$u = gC_o x \tag{2.2}$$

where g is a large positive scalar, the state x is an n-vector and control u is an m-vector. We assume that the pair (A_o, B_o) is controllable, matrices A_o, B_o and C_o are known and rank B_o = rank C_o = m. In [21], it is observed that a dynamic loop whose gain tends to infinity causes change in the system order, characteristic of singular perturbation. On the other hand, a typical singularly perturbed linear time-invariant feedback system is

$$\dot{x}_1 = (F_{11} + F_1H_1)x_1 + (F_{12} + G_1H_2)x_2 \tag{2.3}$$

$$\mu\dot{x}_2 = (F_{21} + G_2H_1)x_1 + (F_{22} + G_2H_2)x_2 \tag{2.4}$$

where μ is a small positive scalar. Clearly by defining

$$u = \frac{1}{\mu}[F_{21} + G_2H_1)x_1 + F_{22} + G_2H_2)x_2] \tag{2.5}$$

to be an artificial high gain feedback, the singularly perturbed system (2.3), (2.4) is equivalent to a high gain feedback system (2.3), (2.5) and

$$\dot{x}_2 = u \tag{2.6}$$

where the gain factor $g = \frac{1}{\mu}$ and $g \to \infty$ as $\mu \to 0$. Motivated by this analogy, we use the singular perturbation methodology to examine the behavior of

of high gain feedback systems. In particular, we focus on the class of high gain feedback systems which possesses a two time scale property. In the case when all the states are accessible, it is possible to design the feedback matrix K in the high gain feedback loop

$$u = gKx \,. \tag{2.7}$$

A procedure for separate placement of fast and slow modes and a decomposition of near optimal high gain regulator problems are presented.

2.1. Fast and Slow Modes

The high gain feedback system (2.1)-(2.2) with $\mu \equiv g^{-1}$ can be written as

$$\mu\dot{x} = (\mu A_o + B_o C_o)x \,. \tag{2.8}$$

In order to apply singular perturbation methods, we transform (8) into the form of system (2.3), (2.4). Let M be such that

$$M\,B_o = \begin{bmatrix} 0 \\ B_2^o \end{bmatrix} \tag{2.9}$$

where B_2^o is a m × m nonsingular matrix. Such a matrix M exists and is the product of elementary row transformations. Let $x' = [x_1' \; x_2']^T \equiv Mx$ and in the new coordinates, (2.8) becomes

$$\dot{x}_1' = A_{11}^o x_1' + A_{12}^o x_2' \tag{2.10}$$

$$\mu\dot{x}_2' = \mu A_{21}^o + B_2^o C_1)x_1' + (\mu A_{22}^o + B_2^o C_2)x_2' \tag{2.11}$$

where C_1, C_2 are m × (n-m) and m × m matrix respectively defined by

$$C_o M^{-1} = [C_1 \quad C_2] \tag{2.12}$$

and matrices A^o_{ij} are defined in Appendix A. System (2.10), (2.11) is now in the same form as (2.3), (2.4). Nevertheless, since it is required in singular perturbation analysis [22,23] that the matrix $F_{22} + G_2H_2$ in (2.4) be a stable matrix, we make the assumption that $\mu A^o_{22} + B^o_2 C_2$ is a stable matrix. For μ sufficiently small, that is, the gain y sufficiently large, this assumption is equivalent to $\mathrm{Re}\lambda(B^o_2 C_2) < 0$. Since B^o_2 is nonsingular, this requires C_2 to be nonsingular. From the identity $C_oB_o = C_2B^o_2$, this assumption becomes

$$\mathrm{Re}\,\lambda(C_oB_o) < 0 \ . \qquad (2.13)$$

The following lemma expresses assumption (2.13) in terms of the matrix B_oC_o in (2.8).

Lemma 2.1:

Assumption (2.13) is equivalent to

$$\mathrm{rank}\ B_oC_o = \mathrm{rank}\ C_oB_o = m \qquad (2.14)$$

and the nonzero eigenvalues of B_oC_o have negative real parts,

$$\mathrm{Re}\,\lambda\,(B_oC_o) < 0,\ i=1,\ \ldots,\ m \ . \qquad (2.15)$$

Proof: Let λ be an eigenvalue of C_oB_o, then there exists a vector v such that

$$C_oB_o v = \lambda v \ . \qquad (2.16)$$

Multiplying to the left with B_o on both sides of (2.16), we have

$$B_oC_o w = \lambda w$$

where $w \equiv B_ov$. Thus λ is a nonzero eigenvalue of B_oC_o. Thus (2.13) is equivalent to (2.14) and (2.15).

Under the assumptions (2.14), (2.15), it will now be shown that the motion of (2.8) consists of a fast transient to an $O(\mu)$ neighborhood of the subspace $C_o x = 0$, followed by a slow motion in this neighborhood. Let $\tilde{x} = [\tilde{x}_1\ \tilde{x}_2]^T \equiv Tx$ where

$$T = \begin{bmatrix} I_{n-m} & 0 \\ C_1 & C_2 \end{bmatrix} M \equiv \Gamma M \ . \qquad (2.17)$$

We note that $\tilde{x}_1 = x_1'$ and $\tilde{x}_2 = C_o x$. It is shown in Appendix A that the meaning of T is that it decompress the state space of x into the null space of C_o, $\mathcal{N}(C_o)$, and the range of B_o, $\mathcal{R}(B_o)$. Then x can be written as

$$x = N\tilde{x}_1 + B_o(C_oB_o)^{-1}\tilde{x}_2 \qquad (2.18)$$

where the columns of the $n \times (n - r)$ matrix N span the null space of C_o, that is,

$$C_oN = 0 \qquad (2.19)$$

and

$$\tilde{x}_1 = M_1 x \qquad (2.20)$$

where matrix M_1 is the first $n - m$ rows of M, that is

$$M_1B_o = 0 \ . \qquad (2.21)$$

Such decompositions also appear in [6,7]. The system (2.8) is thus transformed into

$$\dot{\tilde{x}}_1 = F_{11}\tilde{x}_1 + F_{12}\tilde{x}_2 \qquad (2.22)$$

$$\mu\dot{\tilde{x}}_2 = \mu H_1\tilde{x}_1 + (\mu H_2 + C_oB_o)\tilde{x}_2 \qquad (2.23)$$

where H_i, F_{ij} are defined in Appendix A. The block triangularization [24,25] which is simpler than Jordan transformation used in [6,7] is now applied to system (2.22), (2.23) to exhibit the two time

scale property of high gain feedback system (2.8). We introduce the "fast" variable

$$\eta = \tilde{x}_2 + \mu L \tilde{x}_1 \tag{2.24}$$

where

$$L = L_o + \mu G \tag{2.25}$$

$$L_o \equiv (C_o B_o + \mu H_2)^{-1} H_1 \tag{2.26}$$

and a recursive formula for calculating G is given in Appendix B. The resulting separation of fast and slow modes is now summarized in the following theorem.

Theorem 2.1:

Let the upper bound μ_c for μ (that is a lower bound g_c for g) be given in (B3). *If the high gain feedback system* (2.8) satisfies assumptions (2.14) and (2.15) and if

$$0 \leq \mu \leq \mu_c \tag{2.27}$$

then (2.8) is equivalent to the block triangular system

$$\dot{\tilde{x}}_1 = [F_{11} - \mu F_{12} L]\tilde{x}_1 + F_{12}\eta \tag{2.28}$$

$$\mu\dot{\eta} = [C_o B_o + \mu H_2 + \mu^2 L F_{12}]\eta \tag{2.29}$$

and hence its eigenvalues are approximately.

$$\lambda_i^f = \frac{1}{\mu}[\lambda_i(C_o B_o) + 0(\mu)] \qquad i=1, \ldots, m \tag{2.30}$$

$$\lambda_j^s = \lambda_j(F_{11}) + 0(\mu)] \qquad j=1, \ldots, n-m . \tag{2.31}$$

More, for sufficiently high gain, that is, for μ sufficiently small, the fast subsystem (2.29) is asymtotically stable.

Proof: The proof follows from (2.17)-(2.26) and the existence of G is established in Appendix B. The asymototic stability of (2.29) follows from Assumption (2.13).

The two time scale property of high gain feedback system (2.8) is exemplified by the presence of $O(\frac{1}{\mu})$ large eigenvalues λ_i^f and the eigenvalues λ_j^s which are O(1). The fast variable η decays exponentially in the "stretched" time scale t/μ and for $t \geq t_s$, it is $O(e^{-g\bar{\lambda}t_s})$ where $\bar{\lambda} = |\max \operatorname{Re} \lambda(C_oB_o)|$. Applying lemma C1, the slow variable $\tilde{x}_1$ can be approximated by

$$\tilde{x}_1(t) = e^{F_{11}t}\tilde{x}_1(0) + O(\mu) . \tag{2.32}$$

Hence, the decomposition (2.18) of the original state space corresponds to the separation of time scales and x(t) is approximated by

$$x(t) = N e^{F_{11}t}\tilde{x}_1(0) + B_o(C_oB_o)^{-1}e^{C_oB_o\frac{t}{\mu}}\eta(0) + O(\mu) \tag{2.33}$$

which proves our earlier assertion that, in the limit, the fast transient occurs in $R(B_o)$ and the slow motion is confined to $N(C_o)$. For $t \geq t_s$.

$$\tilde{x}_2(t) \equiv C_ox(t) = \mu L_o\tilde{x}_1(t) + O(\mu^2) . \tag{2.34}$$

If $z = C_ox$ is defined as the regulated variables of system (1) to be forced to zero, that is, a "zeroing the output" problem [26] is considered where z is the output, then from (2.34), we see that z(t) is reduced to an $O(\mu^2)$ quantity after $t \geq t_s$.

We have shown that under the assumption (2.14) and (2.15), the high gain feedback system (2.8) possesses a two time scale property. The recent study of multivariable root locus asymptotes [7] indicates that for more general structures of B_o and C_o, instability of the closed loop

system can result from high gain feedback. For example, in single input system, (2.14) means that the inner product of the input and output vector $c_o^T b_o$ be nonzero. In [27], singular perturbation methods are used to analyze the behavior of the single high gain loop system. It is shown that when $c_o^T b_o = 0$, the closed loop system contains high frequency oscillation modes. If, in addition, $c_o^T A_o^j b_o = 0$ for some $j > 0$, then the system becomes unstable. Thus, we focus on high gain feedback systems which satisfy assumptions (2.14) and (2.15). Clearly, if all the states are accessible, then there always exists feedback gain matrices such that (2.14) and (2.15) holds.

2.2. Root Locus Asymptotes and Transmission Zeros

In Theorem 2.1, we have revealed the asymptotic behavior of the eigenvalues of the high gain feedback system (2.8) as gain tends to infinity. For single input system, the asymptotic behavior of the eigenvalues can be determined by the well known root locus method [5] which provides graphical estimates of the closed loop eigenvalues as a function of the loop gain if the open loop transfer function is known. The finite eigenvalues of the closed loop system coincide with the zeros of the open loop transfer function while the large eigenvalues approach the root locus asymptotes in the limit as gain tends to infinity. In [7], some results of the multivariable root locus asymptotes have been obtained. The relationship of the eigenvalues of a high gain feedback system with the zeros of the open loop system is established in [5]. These zeros are specifically termed transmission zeros since in general they are quite different from the zeros of the various transfer functions relating the inputs and outputs of the system. Later, we will interpret

(2.30) and (2.31) in these terms. First, we define transmission zeros.

Definition 1: [5]

Given the system (2.1) with input u and output

$$y = C_o x \ , \qquad (2.35)$$

where y is an r-vector, the transmission zeros of (2.1), (2.35) (or in short the triple (C_o, A_o, B_o)) are defined to be the set of complex number σ which satisfy the following inequality

$$\operatorname{rank} \begin{bmatrix} A_o - \sigma I & B_o \\ C_o & 0 \end{bmatrix} < n + \min(r,m) \ . \qquad (2.36)$$

This is one of the many definitions of transmission zeros given in [5]. An equivalent definition for the case when $m = r$ is given as follows.

Definition 2: [5,48]

Suppose the pair (A_o, B_o) is controllable, the pair (A_o, C_o) is obserable and $m = r$, then the transmission zeros of the triplet (C_o, A_o, B_o) are the zeros of the polyominal

$$\det(sI - A_o)\det[C_o(sI - A_o)^{-1}B_o] = 0 \ . \qquad (2.37)$$

We note that if $m = r = 1$, then the zeros of (2.37) are in fact the zeros of the open loop transfer function.

From (2.30), it can be seen that as $g \to \infty$, $\mu \to 0$, the eigenvalues λ_i^f tend to infinity along the asymptotes defined by the directions of $\lambda_i(C_oB_o)$ which are the root locus asymptotes obtained in [7]. For large but finite values of gain g, the eigenvalues λ_i^f can be computed from

neglecting the $O(\mu^2)$ terms in 2.29).

To connect the slow eigenvalues in (2.31) with transmission zeros, we need the next theorem proved in [5].

Theorem 2.2: [5]

Let $m = r$ and high gain output feedback $u = gy$ be applied to (2.1), then the finite eigenvalues of the matrix $A_o + gB_oC_o$ coincide with the transmission zeros of the triple (C_o,A_o,B_o).
Thus, the limits $\lambda_j(F_{11})$ of the eigenvalues λ_j^s of (2.31) are the transmission zeros of the triple (C_o,A_o,B_o). For large but finite values of gain g, the eigenvalues λ_j^s can be approximated to any order of μ from the matrix $F_{11} - \mu^2F_{12}G$. Hence, we conclude that for the high gain feedback system (2.1), (2.2) which satisfies conditions (2.14) and (2.15), Theorem 2.1 encompasses and extends the results on root locus asymptotes and transmission zeros of multivariable systems in [5 ,6 , 7]. Moreover, Theorem 2.1 and Appendix A provide the following procedure for calculation of transmission zeros:

Find the matrix $M^T = [M_1^T\ M_2^T]$ in (2.9) and its inverse $M^{-1} = [S_1\ S_2]$ where M_2^T and S_2 are $n \times m$ matrices. This inverse is easy to compute since M is a product of elementary row transformations. Then form $C_oM^{-1} = [C_1\ C_2]$ as in (2.12). Compute C_2^{-1} and the eigenvalues of

$$F_{11} = M_1A_o(S_1 - S_2C_2^{-1}C_1) \tag{2.38}$$

which are the transmission zeros of the triple (C_o,A_o,B_o).

The derivation of (2.38) is given in Appendix A where it is also shown that the columns of the matrix $S_1 - S_2C_2^{-1}C_1$ span the null space of

C_o, that is,

$$\mathcal{R}(N) = \mathcal{R}(S_1 - S_2 C_2^{-1} C_1) \tag{2.39}$$

where N is defined in (2.19). Thus, our matrix F_{11} coincides with the "zero matrix" given in [6 , 7, 9] for the specific choice of the basis for $\mathcal{N}(C_o)$. The calculation of the transmission zeros from matrices of the form (2.38) avoid the ill-conditioning due to the presence of the large gain factor g in the direct computation of the eigenvalues of the matrix $A_o + g B_o C_o$ which is the approach used in [5 ,28].

2.3. Separate Pole Placement

In the case when all the states are accessible, we consider the synthesis of high gain state feedback control $u = gKx$ (2.7) such that the closed loop system (2.1), (2.7) possesses desirable properties. In this section, we are concerned with the synthesis of matrix K such that

$$\text{rank } K B_o = \text{rank } B_o K = m \tag{2.40}$$

and the fast and slow eigenvalues of the high gain state feedback system (2.1), (2.7) are placed at desirable locations. By substituting K for C_o in (2.2), Theorem 2.1 shows that the fast and slow eigenvalues are approximated by

$$\frac{1}{\mu}[\lambda_i(K B_o) + O(\mu)] \qquad i=1, \ldots, m \tag{2.41}$$

and

$$\lambda_j(A_{11}^o - A_{12}^o K_2^{-1} K_1) + O(\mu) \qquad j=1, \ldots, n-m \tag{2.42}$$

respectively where the $m \times (n-m)$ matrix K_1 and then $m \times m$ matrix K_2 are defined by

$$KM^{-1} = [K_1 \quad K_2] . \tag{2.43}$$

The invertibility of K_2 is a consequence of (2.40). Let the complex numbers q_j, $j=1, \ldots, n-m$ and $\mu^{-1}p_i$, $\mathrm{Re}\,p_i < 0$, $i = 1, \ldots, m$ be the prescribed locations of the slow and fast eigenvalues respectively, then the specifications for the synthesis of matrix K become

$$\lambda_i(KB_o) = p_i, \qquad i=1, \ldots, m \tag{2.44}$$

and

$$\lambda_j(A^o_{11} - A^o_{12}K_2^{-1}K_1) = q_j \qquad j=1, \ldots, n-m.. \tag{2.45}$$

We first develop a design procedure motivated by the results of [29]. We then present a complete separate design of the fast and slow modes.

Lemma 2.2:

Let K_ℓ be an $m \times m$ nonsingular matrix and K_z be an $m \times n$ matrix then there exists matrix K of the form

$$K = K_\ell K_z \tag{2.46}$$

such that (2.44) and (2.45) are satisfied.

Proof: Theorem 2 shows that $\lambda_j(A^o_{11} - A^o_{12}K_2^{-1}K_1)$ are the transmission zeros of the triple (K, A_o, B_o). It follows from [29,30] that there exists a matrix K_z such that the transmission zeros of (K_z, A_o, B_o) are q_j, $j=1, \ldots, n-m$. This matrix K_z is not unique and it is suggested in [30] that K_z be determined by choosing n-m elements of K_z such that the following linear equations

$$\det[K_z(q_jI-A_o)^{-1}B_o] = 0, \qquad j=1, \ldots, n-m . \tag{2.47}$$

or alternatively the linear equations [31]

$$\det\begin{bmatrix} A_o - q_jI & B_o \\ K_z & 0 \end{bmatrix} = 0, \qquad j=1, \ldots, n-m \tag{2.48}$$

are satisfied. We note that (2.48) defines q_j as the transmission zeros of (K_z, A_o, B_o) as in (2.36). The placement of the n-m transmission zeros implies that K_zB_o is nonsingular [6]. Thus, K_z satisfies (2.40) and the fast eigenvalues in the limit are $\lambda_i(K_zB_o)$. Since $(K_zB_o)^{-1}$ exists, there exists matrix K_ℓ such that

$$\lambda_i(K_\ell K_zB_o) = p_i, \qquad i=1, \ldots, m . \tag{2.49}$$

The transmission zeros of (K_z, A_o, B_o) and $(K_\ell K_z, A_o, B_o)$ are the same since

$$\text{rank}\begin{bmatrix} A_o - \sigma I & B_o \\ K_\ell K_z & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} A_o - \sigma I & B_o \\ K_z & 0 \end{bmatrix} . \tag{2.50}$$

This proves the lemma.

A procedure to place the slow and fast eigenvalues is developed in the proof of this lemma. The design consists of two stages, first the slow eigenvalues are placed by solving for K_z in (2.47) or (2.48). Then, the fast eigenvalues are placed by solving for K_ℓ in (2.49). This design procedure encompasses the design in [29] where it is demanded that KB_o be negative definite which is equivalent to restricting p_i in (2.44) to be real numbers.

We note that the placement of the slow eigenvalues in this design is achieved by finding an appropriate "output" matrix K so that the triple (K, A_o, B_o) has transmission zeros coincide with q_j. Since we have shown that the transmission zeros coincide with $\lambda_j(A_{11}^o - A_{12}^o K_2^{-1} K_1)$, it is reasonable to expect that the placement of the slow eigenvalues can be achieved by solving a pole placement problem. To show that this is indeed the case, let M be the transformation defined by (2.9) and $x = [x_1' \; x_2']^T \equiv Mx$. Then, (2.1) is transformed into

$$\dot{x}_1' = A_{11}^o x_1' + A_{12}^o x_2' \tag{2.51}$$

$$x_2' = A_{21}^o x_1' + A_{22}^o x_2' + B_2^o u \quad . \tag{2.52}$$

Lemma 2.3:

If the pair (A_o, B_o) is controllable (stabilizable), then the pair (A_{11}^o, A_{12}^o) is controllable (stabilizable).

Proof: The controllability of (A_o, B_o) implies [31]

$$\text{rank}\begin{bmatrix} \sigma I - A_{11}^o & A_{12}^o & 0 \\ A_{21}^o & \sigma I - A_{22}^o & B_2^o \end{bmatrix} = n \tag{2.53}$$

and, since rank $B_2^o = m$,

$$\text{rank}\,[\,\sigma I - A_{11}^o \quad A_{12}^o\,] = n - m \tag{2.54}$$

for all complex number σ. Thus, (A_{11}^o, A_{12}^o) is controllable. When (A_o, B_o) is only stabilizable, consider $u = v - B_2^{o^{-1}} A_{21} x_1$. Since (A_{22}, B_2) is controllable, rank $[\sigma I - A_{11}^o \quad A_{12}^o] < n\text{-}m$ and all uncontrollable but stable

eigenvalues of A_o are eigenvalues of A^o_{11}. Hence (A^o_{11}, A^o_{12}) is stabilizable. Since by assumption, (A_o, B_o) is controllable from Lemma 2.3, there exists an $(n-m)\times m$ matrix K_s such that

$$\lambda_j(A^o_{11} - A^o_{12}K_s) = q_j, \quad j=1, \ldots, n-m . \tag{2.55}$$

By defining $K_s = K_2^{-1}K_1$, we have shown that the slow eigenvalues can be placed by solving a (n-m)-dimensional pole placement problem. By observing that

$$\lambda_i(KB_o) = \lambda_i(K_2B^o_2) = \lambda_i(B^o_2K_2) , \tag{2.56}$$

the fast eigenvalues are placed by solving a m-dimensional pole placement problem. By defining $K_f \equiv K_2$ and letting $K_1 = K_fK_s$, we recognize that the matrix K_s and K_f which place the slow and fast eigenvalues respectively are designed independently.

<u>Theorem 2.3</u>:

Let (A_o, B_o) be a controllable pair and let the $(n-m)\times m$ matrix K_s and $m\times m$ nonsingular matrix K_f be such that (2.55) and

$$\lambda_i(K_fB^o_2) = p_i, \quad i=1, \ldots, m \tag{2.57}$$

holds. Then, the use of the high gain feedback

$$u = g[K_fK_s x'_1 + K_f x'_2] \tag{2.58}$$

places the eigenvalues of the system (2.51), (2.52) to $q_j + 0(\frac{1}{g})$ and $g[p_i + 0(\frac{1}{g})]$.

<u>Proof</u>: Substituting (2.58) in (2.51), (2.52) and applying the transformation $\tilde{x} = \Gamma x'$ where Γ is as in (2.17) yields

$$\dot{\tilde{x}}_1 = (A^o_{11} - A^o_{12}K_s)\tilde{x}_1 + A^o_{12}K_f^{-1}\tilde{x}_2 \tag{2.59}$$

$$\begin{aligned}\mu\dot{\tilde{x}}_2 &= \mu K_f[K_s A^o_{11} - A^o_{22}K_s - K_s A^o_{12}K_s + A^o_{12}]\tilde{x}_1 \\ &+ K_f B^o_2[I + \mu B^o_2{}^{-1}(K_s + A^o_{22}K_f^{-1})]\tilde{x}_2 \ .\end{aligned} \tag{2.60}$$

The proof follows from Lemma 2.1 and Theorem 2.1 and (2.56).

From the above theorem, a procedure for completely separated placement of the slow and fast eigenvalues can be summarized: design K_s to place the eigenvalue of $A^o_{11} - A^o_{12}K_s$ and design K_f to place the eigenvalues of $B^o_2 K_f$ then form the high gain feedback control as in (2.58).

As a simple illustration consider a system in the form of (2.51), (2.52),

$$\dot{x} = \begin{bmatrix} 3 & 1 & 1 \\ -6 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} u \quad . \tag{2.61}$$

First, suppose that a high gain feedback control is to be found in place of the slow eigenvalue near $\lambda_1 = -3$ and the fast eigenvalues near $\lambda_{2,3} = g(-1 \pm j1)$. We solve the two lower order pole placement problems (2.55), (2.57) and obtain

$$K_s = \begin{bmatrix} 6 \\ 0 \end{bmatrix} , \quad K_f = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} . \tag{2.62}$$

The high gain feedback (2.58) is

$$u = g \begin{bmatrix} 0 & 0 & 1 \\ -12 & -2 & 0 \end{bmatrix} x \ . \tag{2.63}$$

Second, consider the high gain feedback system (2.61), (2.63) with (2.63) written as $u = g\,C_o x$ and apply Theorem 2.1. The closed loop eigenvalues are computed approximately by neglecting the μ^2 terms in (2.28), (2.29), that is,

$$\lambda_1 = \lambda(F_{11} - \mu F_{12} L_o) = \lambda(-3 + 30\mu) = -3 + 30\,\mu \qquad (2.64)$$

$$\lambda_{2,3} = \frac{1}{\mu}\,\lambda(C_o B_o + H_2) = \frac{1}{\mu}\,\lambda\left(\begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} + \mu \begin{bmatrix} 3 & 0 \\ -12 & 7 \end{bmatrix}\right)$$

$$= \frac{1}{\mu}\,(-1 + 5\mu \pm j\sqrt{1 + 16\mu - 4\mu^2}) \quad . \qquad (2.65)$$

The upper bound in (2.27) is $\mu_c = 0.0074$ corresponding to $g_c = 135.6$. The exact eigenvalues are computed for comparison purpose. Denoting the error between the approximate and exact eigenvalues as ϵ_{app} and between the specified and exact eigenvalues as ϵ_{spc}, the results are summarized by the following table.

Gain g	Exact	Approximate	Specified	ϵ_{app}%	ϵ_{spc}%
100	-2.69324752 -95.1533625 ± ± j107.955057	-2.7 -95 ± ± j107.684725	-3.0 -100 ± j100	< 0.05	< 14
1000	-2.96991225 -995.015044 + + j1007.99596	-2.97 -995 ± ± j1007.96627	-3.0 -1000 ± j1000	< 0.003	≈ 1

From (2.64), (2.65) it is observed that as $\mu \to 0$, $\lambda_{1,2,3}$ tend to the specified values. For design purposes gain g does not have to be high, since with $g = 100$ the accuracy of 14% in eigenvalue location is often acceptable.

2.4. Near Optimum High Gain Regulators

High gain feedback systems can also result for the optimization of system (2.1) with respect to a quadratic performance index having small penalty on the control variables u,

$$J = \frac{1}{2}\int_0^\infty [x^T Q_o x + \mu^2 u^T R u] dt \tag{2.66}$$

where Q_o is symmetric positive semidefinite, R is symmetric positive definite and μ is a small positive scalar parameter. This class of problems belong to the so called "cheap control" problems which have been considered in [11,12,13,32,33]. Detailed results exist [11] for the case when

$$B_o^T Q_o B_o > 0 \ . \tag{2.67}$$

Under this condition, the resulting optimal high gain feedback system satisfies the assumption (2.14). Thus the analysis of section 2.1 is applicable to reveal the two time scale property of the optimal state regulator developed in [11]. We assume that the transformation M has been applied to (2.1) and we consider the open loop system (2.51), (2.52). Correspondingly, the weighting matrix on the states becomes $Q = (M^{-1})^T Q_o M^{-1}$ and we denote

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \tag{2.68}$$

where Q_{11} and Q_{22} are $(n-m) \times (n-m)$ and $m \times m$ matrices respectively. The condition (2.67) becomes

$$B_2^{o^T} Q_{22} B_2^o > 0 \ . \tag{2.69}$$

Since B_2^o is nonsingular, (2.69) implies Q_{22} is nonsingular. We now summarize the results in [11] with the next theorem.

Theorem 2.4:

Suppose condition (2.69) holds, the pair $(A_{11}^o - A_{12}^o Q_{22}^{-1} Q_{12}^T, A_{12}^o)$ is stabilizable and the pair $(\mathcal{D}, A_{11}^o - A_{12}^o Q_{22}^{-1} Q_{12}^T)$ is detectable where

$$\mathcal{D}^T \mathcal{D} = Q_{11} - Q_{12} Q_{22}^{-1} Q_{12}^T \tag{2.70}$$

then the optimal Riccati solution P possesses a power series expansion for $\mu = 0$,

$$P = \sum_{\ell=0}^{\infty} \begin{bmatrix} P_{11\ell}\mu^{\ell} & P_{12\ell}\mu^{\ell+1} \\ P_{12\ell}^T \mu^{\ell+1} & P_{22\ell}\mu^{\ell+1} \end{bmatrix} \tag{2.71}$$

where $P_{11\ell}$ and $P_{22\ell}$ are $(n-m) \times (n-m)$ and $m \times m$ matrices respectively, and the zeroth order term P_{110} is the unique symmetric semidefinite solution of the matrix Riccati equation,

$$\begin{aligned} P_{110}(A_{11}^o - A_{12}^o Q_{22}^{-1} Q_{12}^T) + (A_{11}^o - A_{12}^o - A_{12}^o Q_{22}^{-1} Q_{12}^T)^T P_{110} \\ + \mathcal{D}^T \mathcal{D} = P_{110} A_{12}^o A_{12}^o Q_{22}^{-1} A_{12}^{o\,T} P_{110}, \end{aligned} \tag{2.72}$$

P_{220} is a symmetric positive definite matrix,

$$P_{220} = S^{-1/2}[S^{-1/2} Q_{22} S^{-1/2}]^{1/2} S^{-1/2} \tag{2.73}$$

with

$$S = B_2^o R^{-1} B_2^{oT} \tag{2.74}$$

and P_{120} satisfies

$$P_{120} = (P_{110} A_{12}^o + Q_{12}) P_{220}^{-1} S^{-1}. \tag{2.75}$$

Moreover, the optimal feedback control is given by

$$u = -\frac{1}{\mu} R^{-1} B_2^{oT} \left(\sum_{\ell=0}^{\infty} [P_{12\ell}^T x_1' + P_{22\ell} x_2'] \mu^{\ell}\right) \tag{2.76}$$

From Theorem 2.1, we see that the eigenvalues of the optimal high gain feedback system (51), (52), (76) are approximately

$$\lambda_i^f = \frac{1}{\mu} [\lambda_i(-R^{-1} B_2^{oT} P_{220} B_2^o) + O(\mu)], \quad i = 1,\ldots,m \tag{2.77}$$

and

$$\lambda_j^s = \lambda_j(A_{11}^o - A_{12}^o P_{220}^{-1} P_{120}^T) + O(\mu), \quad j = 1,\ldots,n-m \tag{2.78}$$

Since the matrices R and P_{220} are positive definite and B_2^o is non-singular, the matrix $-R^{-1}B_2^{oT}P_{220}B_2^o$ is negative definite. Hence, the root locus asymptotes form an angle of 180° with the positive real axis in the complex plane.

From the fast-slow separation of the pole placement design, it can be expected that a similar decomposition is feasible in the near optimum high gain regulator design. Instead of the full order problem, the following two reduced order regulator problems are solved.

<u>Problem "s"</u>:

Optimize system (2.79) with respect to the performance index (2.80)

$$\dot{x}_s = A_{11}^o x_s + A_{12}^o u_s \tag{2.79}$$

$$J_s = \frac{1}{2} \int_0^{\infty} (x_s^T Q_{11} x_s + 2x_s^T Q_{12} u_s + u_s^T Q_{22} u_s) dt. \tag{2.80}$$

Problem "f":

Optimize system (2.81) with respect to the performance index (2.82)

$$\dot{x}_f = B_2^o u_f \tag{2.81}$$

$$J_f = \frac{1}{2}\int_o^\infty (x_f^T Q_{22} x_f + u_f^T R u_f)dt. \tag{2.82}$$

In (2.79), (2.81), matrices A_{11}^o, A_{12}^o and B_2^o are as in (2.51), (2.52) and x_s is an (n-m)-vector, x_f, u_f and u_s are m-vectors. In (2.80), (2.82), matrices Q_{ij} are as in (2.68) and matrix R is as in (2.66).

Lemma 2.4:

If the pair (A_o, B_o) is stabilizable and

$$(A_{11}^o, \mathcal{D}) \text{ is detectable} \tag{2.83}$$

where $\mathcal{D}$ is defined in (2.70) and if (2.69) is satisfied, then there exists a unique stabilizing solution P_s (which is symmetric positive semidefinite) if the matrix Riccati equation

$$0 = P_s(A_{11}^o - A_{12}^o Q_{22}^{-1} Q_{12}^T) - (A_{11}^o - A_{12}^o Q_{22}^{-1} Q_{12}^T)^T P_s$$

$$+ P_s A_{12}^o Q_{22}^{-1} A_{12}^{oT} P_s - \mathcal{D}^T \mathcal{D}. \tag{2.84}$$

and the optimal control for Problem "s" is

$$u_s = - Q_{22}^{-1} (Q_{12}^T + A_{12}^{oT} P_s) x_s \equiv G_s x_s. \tag{2.85}$$

Proof: By Lemma 2.3, the pair (A_{11}^o, A_{12}^o) is stabilizable and hence $(A_{11}^o - A_{12}^o Q_{22}^{-1} Q_{12}^T, A_{12}^o)$ is stabilizable. Together with (2.83), the existence and uniqueness of P_s is guaranteed [34].
Since B_2^o and Q_{22} are nonsingular, the optimal control for problem "f" is

readily obtained as

$$u_f = -R^{-1}B_2^oP_fx_f \equiv G_fx_f \tag{2.86}$$

where

$$P_f = S^{-1/2}(S^{1/2}Q_{22}S^{1/2})^{1/2}S^{-1/2} > 0. \tag{2.87}$$

A near optimal control u_c for the system (2.51), (2.52) is now comprised of the controls u_s and u_f in the form of the control law (2.58)

$$u_c = g(G_fG_sx_1' + G_fx_2'). \tag{2.88}$$

It is crucial to note that the feedback gain matrices are operating on the state variables x_1' and x_2' of the system (2.51), (2.52) rather than on x_s and x_f. The feedback system (2.51), (2.52), (2.88) satisfies Theorem 2.3 with G_s and G_f playing the roles of K_s and K_f, respectively. Thus, J_s and J_f can be interpreted as the performance specifications for the slow and fast modes.

<u>Theorem 2.5</u>:

Under the conditions of Lemma 2.4, the performance J of the feedback system (2.51), (2.52), (2.88) is $O(\mu^2)$ close to its optimum performance.

<u>Proof</u>: Since (2.84) and (2.87) coincide with (2.72), (2.73), the leading terms P_{110} and P_{220} of the optimal Riccati solution expansion (2.71) are identical to P_s and P_f, respectively. From the identity (2.75) which is equivalent to

$$P_s A^o_{12} + Q_{12} = P_{120} S P_f \tag{2.89}$$

it follows that u_c is $O(\mu)$ close to the optimal control (2.76), that is,

$$u_c = u + O(\mu) \tag{2.90}$$

and the states x of the feedback system (2.51), (2.52), (2.88) is $O(\mu)$ close to the states of the optimal feedback system (2.51), (2.52), (2.76). By Theorem 3 in [35], the performance of (2.51), (2.52), (2.88) is $O(\mu^2)$ near optimal.

The results on cheap control [11] are thus recast in terms of two separate problems which give an $O(\mu^2)$ near optimal solution. In view of Lemma 2.4, we note that the stabilizability assumption in Theorem 2.4 is equivalent to the assumption that (A_o, B_o) be stabilizable which is the standard assumption made in linear quadratic regulator problems. The standard assumption on the detectability of (C, A_o) with $C^T C = Q_o$ is, however, replaced by detectability assumptions (2.83) and (2.69) in cheap control problems which can be interpreted as the detectability of the slow and fast subsystems respectively.

If instead of a state regulator problem, a output regulator problem is to be solved, that is, (2.66) is replaced by

$$J = \frac{1}{2} \int_0^\infty [z^T z + \mu^2 u^T R u] dt \tag{2.91}$$

where the output is denoted by an m-vector, $z = C_o x$. If rank $C_o = m$, and $C_o B_o$ is nonsingular, then by making the substitution $Q_o = C_o^T C_o$ in (2.66), an $O(\mu^2)$ near optimal control can be found by the preceding decomposition procedure. The matrix Q of (2.68) has the form

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} = \begin{bmatrix} C_1^T & C_1^T C_2 \\ C_2^T C_1 & C_2^T C_2 \end{bmatrix} \tag{2.92}$$

where C_1, C_2 are as in (2.12). For Problem "s" and "f", introduce

$$y_s = C_1 x_s \tag{2.93}$$

$$y_f = C_2 x_f \tag{2.94}$$

as the output variables of system (2.79) and (2.81), respectively. Then, these two problems become output regulator problems. For the matrix Q given by (2.92), $\mathcal{D} = 0$ in (2.70). Thus, we have the following lemma.

Lemma 2.5:

If the pair (A_o, B_o) is stabilizable and

$$\mathrm{Re}\lambda(A_{11}^o - A_{12}^o C_2^{-1} C_1) < 0 \tag{2.95}$$

then the unique stabilizing solution P_s of the matrix Riccati equation (2.84) is

$$P_s = 0 \tag{2.96}$$

and the optimal control for Problem "s" is

$$u_s = - C_2^{-1} C_1 x_s \equiv G_s x_s. \tag{2.97}$$

Proof: This lemma is a consequence of Theorem 13 in [36].

The composite control u_c in this case becomes

$$u_c = g(R^{-1} B_2^o P_f C_2^{-1} C_1 x_1' + R^{-1} B_2^o P_f x_2') \tag{2.98}$$

and the slow eigenvalues of the optimal and near optimal feedback systems can be approximated by

$$\lambda_j^s = \lambda_j(A_{11}^o - A_{12}^o C_2^{-1} C_1) + O(\mu), \ j = 1,\ldots,n-m. \tag{2.99}$$

From Theorem 2.2, it is shown that $\lambda_j(A_{11}^o - A_{12}^o C_2^{-1} C_1)$ are the transmission zeros of the triple (C_o, A_o, B_o). Hence, we conclude that if all the transmission zeros of (C_o, A_o, B_o) lie in the open left half complex plane, then the finite eigenvalues of the optimal feedback system tend to these transmission zeros as the small penalty on the control variables $\mu \to 0$. The condition (2.95) is analogous to the condition on transmission zeros for achieving "maximal accuracy" in [12].

CHAPTER 3

HIGH GAIN FEEDBACK FOR SYSTEMS WITH UNCERTAINTIES AND INACCESSIBLE STATES

It is well known from classical control theory that high gain loops are capable of reducing sensitivity of single input control systems with respect to parameter variations and disturbances. In the same spirit, the properties of multivariable linear systems with high gain loops are under investigation in this chapter. The system considered is of the form

$$\dot{x} = (A_o + \delta A)\, x + (B_o + \delta B)u + E_o f \tag{3.1}$$

$$y = (C_o + \delta C)x \tag{3.2}$$

$$z = D_o x \tag{3.3}$$

where the state $x \in R^n$, the control input $u \in R^m$, the measured output $y \in R^r$, the disturbance $f \in R^p$ and the regulated variables $z \in R^\ell$. It is assumed that the nominal values of the system parameters A_o, B_o, C_o and E_o are known and the matrix D_o is also known. Furthermore, the disturbance f is not measurable and the only information about it is that $|\dot{f}| \leq h_1$ and $|f| \leq h_2$ for some positive constants h_1 and h_2.

The following problems are addressed. Given the measurements y(t), under what conditions does a feedback control exist such that desirable attenuation of the effects of disturbances and parameter variations on the regulated variables z(t) is achieved? What are the synthesis procedures for such a feedback control?

The class of feedback controls which we conjecture to have the desirable properties is high gain observer feedback

$$u = gK\hat{x} \tag{3.4}$$

where g is a large scalar. This calls for a full order observer which has the property that the estimation error

$$e' \equiv x - \hat{x} \tag{3.5}$$

between the states x and observer states $\hat{x}$ can be reduced to zero asymtotically in the presence of parameter variations and disturbances. A prospective structure for an observer with such capability is one that itself contains high gain loops, that is,

$$\dot{\hat{x}} = A_o\hat{x} + B_o u - Hv \tag{3.6}$$

$$v = \hat{g}\,(y - C_o\hat{x}) \tag{3.7}$$

where $\hat{g}$ is a large scalar. The closed loop system (3.1) - (3.4), (3.6) - (3.7) contains two groups of high gain loops and the design parameters are the matrices K and H.

The organization of this chapter is as follows. First, we analyze the properties of system (3.1) with high gain state feedback

$$u = gKx \tag{3.8}$$

whre K is a given matrix appropriately designed, for example, by the methods in Sections 2.3, 2.4. A condition for x(t) to have arbitrarily small sensitivities with respect to parameter variations and disturbances

is obtained. Similarly, the analysis is then applied to the high gain observer system (3.6), (3.7) for a given matrix H and a condition for $e(t)$ to have arbitrarily small sensitivities is obtained. Thirdly, conditions are derived for the regulated variables $z(t)$ to have small sensitivities when full state high gain feedback (3.8) is permissible. Then we develop conditions when high gain observer feedback (3.4) is applied. Finally, we look into the synthesis problems of matrices K and H that satisfy the above conditions and at the same time place the eigenvalues of the main loops and the observer loops.

3.1 High Gain State Feedback

The behavior of system (3.1) when high gain state feedback (3.4) is applied is examined. We assume that the feedback matrix K is designed such that the closed loop undisturbed nominal system, that is, (3.1), (3.4) with δA, δB, δC and E_o equal to null matrices of appropriate dimensions, possesses a two time scale property and desirable characteristics. As it is shown in the last chapter, we can assume that the transformation M of (2.9) has been applied to (3.1) and in the new coordinates (3.1) is represented by

$$\dot{x}_1' = (A_{11}^o + \delta A_{11})x_1' + (A_{12}^o + \delta A_{12})x_2' + \delta B_1 u + E_1^o f \qquad (3.9)$$

$$\dot{x}_2' = (A_{21}^o + \delta A_{21})x_1' + (A_{22}^o + \delta A_{22})x_2' + (B_2^o + \delta B_2)u + E_2^o f \quad (3.10)$$

where $x^T = [x_1' \; x_2']^T$, x_1' is an (n-m) vector, x_2' is an m-vector and, B_2^o is a m×m nonsingular matrix. We adopt the convention that matrices with superscript o and prefix δ denote nominal values and parameter

variations, respectively. Let the high gain state feedback (3.4) in these coordinates be

$$u = g[K_f K_s x_1' + K_f x_2'] \tag{3.11}$$

as in (2.58). Our assumption on K then becomes

$$\mathrm{Re}\ \lambda(A_{11}^o - A_{12}^o K_s) < 0 \tag{3.12}$$

and

$$\mathrm{Re}\ \lambda(KB_o) < 0. \tag{3.13}$$

Introducing the variable

$$\omega = K_f K_s x_1' + K_f x_2' \tag{3.14}$$

the condition for x(t) to have arbitrarily small sensitivities with respect to parameter variations and disturbances is now stated in the following theorem.

Theorem 3.1:

The solution of the closed loop system (3.9)-(3.11) can be approximated by

$$x_1(t) = e^{(A_{11}^o - A_{12}^o K_s)t} x_1(0) + 0(g^{-1}) \tag{3.15}$$

$$\omega(t) = e^{gK(B_o + \delta B)t} \omega(0) + 0(g^{-1}) \tag{3.16}$$

if and only if

$$\mathcal{R}(E_o) \subseteq \mathcal{R}(B_o), \tag{3.17}$$

$$\mathcal{R}(\delta A) \subseteq \mathcal{R}(B_o) \tag{3.18}$$

$$\mathcal{R}(\delta B) \subseteq \mathcal{R}(B_o) \tag{3.19}$$

and

$$\mathrm{Re}\lambda[K(B_o + \delta B)] < 0. \tag{3.20}$$

Moreover, let $\sigma_s = \max \mathrm{Re}\lambda[K(B_o + \delta B)]$ and $t_s = |\sigma_s|^{-1}\mu\ell n\mu$. Then for all $t \geq t_s$, $\omega(t)$ is reduced to $0(g^{-1})$, that is, $|\omega(t)| = 0(g^{-1})$.

<u>Proof</u>: In terms of the new variable ω, the closed loop system (3.9)-(3.11) has the form

$$\dot{x}_1' = (Z_1^o + \delta Z_1)x_1' + (Z_2^o + \delta Z_2 + g\delta B_1)\omega + E_1^o f \tag{3.21}$$

$$\dot{\omega} = (F_1^o + \delta F_1)x_1' + [F_2^o + \delta F_2 + gK(B_o + \delta B)]\omega + KE_o f \tag{3.22}$$

where

$$Z_1^o = A_{11}^o - A_{12}^o K_s \tag{3.23}$$

$$Z_2^o = A_{12}^o K_f^{-1} \tag{3.24}$$

$$F_1^o = K_f K_s A_{11}^o + K_f A_{21}^o - F_2^o K_f K_s \tag{3.25}$$

$$F_2^o = (K_f A_{22}^o + K_f K_s A_{12}^o)K_f^{-1} \tag{3.26}$$

and the matrices δZ_i, δF_i are defined as in (3.23), (3.24) and (3.25), (3.26) with A_{ij}^o replaced by δA_{ij}. We first prove sufficiency. Conditions (3.17)-(3.19) imply that $\delta B_1 = 0$, $E_1^o = 0$, $\delta A_{11}^o = 0$ and $\delta A_{12}^o = 0$. Consequently, $\delta Z_1 = 0$ and $\delta Z_2 = 0$. Using Lemma C2, we obtain (3.15).

Applying the variation of parameters formula on (3.22), we have

$$\omega(t) = e^{K(B_o + \delta B)\frac{t}{\mu}}\omega(0) + \int_0^t e^{K(B_o + \delta B)\frac{(t-\tau)}{\mu}} KE_o f(\tau)d\tau + O(\mu). \tag{3.27}$$

Integrating the integral by parts, (3.27) becomes

$$\omega(t) = e^{K(B_o + \delta B)\frac{t}{\mu}}\omega(0) + \tag{3.28}$$

$$\mu\int_0^t [K(B_o + \delta B)]^{-1} e^{K(B_o+\delta B)\frac{(t-\tau)}{\mu}} KE_o\dot{f}(\tau)d\tau + O(\mu)$$

Since $|\dot{f}| \leq h_1$ by assumption (3.28) becomes (3.16). Necessity can be proved by contradiction. Suppose condition (3.17) is violated, then $E_1^o \neq 0$ and

$$x_1'(t) = e^{Z_1^o t} x_1'(0) + \int_o^t e^{Z_1^o(t-\tau)} E_1^o f(\tau)d\tau + O(\mu). \tag{3.29}$$

Clearly, the integral in (3.29) is not $O(\mu)$. Now, suppose conditions (3.18)(3.19) are violated, then $\delta B_1 \neq 0$, $\delta Z_1 \neq 0$ and $\delta Z_2 \neq 0$. Lemma C3 gives

$$x_1'(t) = e^{(Z_1^o + \delta Z_1 + \delta B_1 L)t}[x_1'(0) - \delta B_1[K(B_o + \delta B)]^{-1}\omega(0)]$$

$$+ \delta B_1[K(B_o + \delta B)]^{-1}\omega(t) - \int_0^t e^{(Z_1^o + \delta Z_1 + \delta B_1 L)(t-\tau)}.$$

$$\delta B_1[K(B_o+\delta B)]^{-1}KE_o f(\tau)d\tau + O(\mu) \tag{3.30}$$

where $L \equiv [K(B_o + \delta B)]^{-1}(F_1^o + \delta F_1)$. This completes the proof.

The following corollary is evident from the proof.

Corollary 3.1:

The solution of the systems (3.9)-(3.11) can be approximated by

$$x_1'(t) = e^{(Z_1^o + \delta Z_1 + \delta B_1 L)t}[x_1'(0) - \delta B_1(K(B_o + \delta B))^{-1}\omega(0)]$$

$$+ \delta B_1(K(B_o + \delta B))^{-1}\omega(t) + 0(\mu) \tag{3.31}$$

and (3.16) if and only if

$$\mathcal{R}[(I - \delta B(K(B_o + \delta B))^{-1}K)E_o] \subseteq \mathcal{R}(B_o) \tag{3.32}$$

For sufficiently small μ, this system is asymtotically stable if and only if

$$\mathrm{Re}\lambda[Z_1^o + \delta Z_1 + \delta B_1 L] < 0 \tag{3.33}$$

and condition (3.20) holds where the eigenvalues of the matrix $Z_1^o + \delta Z_1 + \delta B_1 L$ are the transmission zeros of the triple $(K, A_o + \delta A, B_o + \delta B)$.

The above theorem and its corollary show that the sensitivities of the high gain state feedback system (3.9)-(3.11) with respect to parameter variations δA, δB and disturbances f depend on the ranges of δA, δB and E_o. When the ranges of these matrices all lie inside the range of B_o, the motion $x_1'(t)$ has $0(\mu)$ sensitivities and the influences of disturbance and parameter variations to $\omega(t)$ for $t \geq t_s$ are also $0(\mu)$. The stability of the fast transient in $\omega(t)$ in the "stretched" time scale gt depends on δB. By an earlier assumption (3.13), there exists δB with $\|\delta B\|$ sufficiently small such that the fast transient

decays exponentially. Thus, we say, the stability of the fast motion $\omega(t)$ is robust in the sense of (3.20). If the range conditions (3.17)-(3.19) are not satisfied but condition (3.32) holds, then the sensitivity of the motion $x_1'(t)$ with respect to f is $O(\mu)$. From (3.31), the motion $x_1'(t)$ is no longer "slow" and is influenced by the system parameter variations δA and δB. Furthermore, the stability of system (3.9)-(3.11) is only robust in the sense of (3.20) and (3.33).

As an illustration, consider the 3rd order system given by (2.61) in the Section 2.3. Suppose this is the nominal undisturbed system. If the columns of the disturbance matrix E_o and the parameter variation matrix δA, δB are of the form $[0 \quad \delta_1 + \delta_2 \quad \delta_1]^T$ for any values of δ_1 and δ_2, then the range conditions (3.17)-(3.19) are satisfied. If the first element of any of the columns is nonzero, then (3.17)-(3.19) do not hold.

3.2 High Gain Observer

The objective of the high gain observer (3.6),(3.7) is to reduce the estimation error e' to zero. Expressing (3.6),(3.7) in terms of e', that is,

$$\dot{e}' = A_o e' + Hv + \delta Ax + \delta Bu + E_o f \tag{3.34}$$

$$v = \hat{g}(C_o e' + \delta Cx) \tag{3.35}$$

it is clear that (3.34), (3.35) represent a high gain feedback system with disturbances x, u and f. The design parameter is the "input" matrix H while the feedback matrix C_o is specified. Thus, the high gain observer design becomes the design of matrix H such that the system (3.34),(3.35)

is asymptotically stable. This design problem can be viewed as the dual problem of the high gain state feedback design problem solved in Section 2.3. This duality is set forth by the following theorem.

Theorem 3.2:

Let (A_o, C_o) be an observable pair, then there exists a matrix H such that the slow and fast eigenvalues of the high gain feedback system

$$\dot{e}' = (A_o + \hat{g}HC_o)e' \tag{3.36}$$

are placed to $q_j + 0(\hat{g}^{-1})$, $j = 1,\ldots,n-r$ and $\hat{g}[p_i + 0(\hat{g}^{-1})]$ $i = 1,\ldots,r$ where q_j and $\hat{g}p_i$ are the prescribed locations of the slow and fast eigenvalues, respectively.

Proof: By the Duality Theorem of Kalman, (A_o^T, C_o^T) is a controllable pair. Since $\lambda(A_o^T + \hat{g}C_o^T H^T) = \lambda(A_o + gHC_o)$, the pole placement problem for (3.36) is equivalent to finding the feedback matrix H^T for the pair (A_o^T, C_o^T). By Theorem 2.3, the theorem is proved.

In the rest of this section, we assume that the design of matrix H by the methods presented in Section 2.3 has been carried out. Since system (3.34), (3.35) is similar to the high gain state feedback system (3.9)-(3.11), it is to be expected that the conditions for $e'(t)$ to be insensitive to the disturbances involve the ranges of δA, δB, E_o and H. Before we state the theorem, it is convenient to represent the system (3.34), (3.35) in the new coordinates defined by

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \hat{M}e' \tag{3.37}$$

where e_1 is an (n-r)-vector, e_2 is an r-vector and $\hat{M}$ is defined by

$$\hat{M}H = \begin{bmatrix} 0 \\ H_2 \end{bmatrix} \tag{3.38}$$

and H_2 is an $r \times r$ nonsingular matrix. Thus we have

$$\dot{e}_1 = \hat{A}^o_{11}e_1 + \hat{A}^o_{12}e_2 + \delta\hat{A}_1x + \delta\hat{B}_1u + \hat{E}^o_1f \tag{3.39}$$

$$\dot{e}_2 = \hat{A}_{21}^{\,o}\, e_1 + \hat{A}^o_{22}e_2 + H_2v + \delta\hat{A}_2x + \delta\hat{B}_2u + \hat{E}^o_2f \tag{3.40}$$

$$v = \hat{g}[C^o_1e_1 + C^o_2e_2] + \hat{g}\delta Cx \tag{3.41}$$

where C^o_2 is an $r \times r$ nonsingular matrix. Our assumption on H is then

$$\mathrm{Re}\lambda\ (\hat{A}^o_{11} - \hat{A}^o_{12}C^{o^{-1}}_2\ C^o_1) < \ 0 \tag{3.42}$$

and

$$\mathrm{Re}\lambda\ (C_oH) < 0 \tag{3.43}$$

The theorem is stated in terms of the variable η

$$\eta = C^o_1e_1 + C^o_2e_2. \tag{3.44}$$

<u>Theorem 3.3</u>:

The solution of the closed loop system (3.37), (3.38) can be approximated by

$$e_1(t) = e^{(\hat{A}^o_{11} - \hat{A}^o_{12}C^{o^{-1}}_2\ C^o_1)t}\, e_1(0) + 0(\hat{g}^{-1}) \tag{3.45}$$

$$\eta(t) = e^{\hat{g}C_oHt}\, \eta(0) + 0(\hat{g}^{-1}) \tag{3.46}$$

if and only if

$$R(\delta A) \subseteq R(H) \tag{3.47}$$

$$R(\delta B) \subseteq R(H) \tag{3.48}$$

$$R(E_o) \subseteq R(H) \tag{3.49}$$

and

$$\delta C = 0. \tag{3.50}$$

Moreover, let $\hat{\sigma}_s = \max \mathrm{Re}\lambda(C_o H)$ and $t_s = |\sigma_s|^{-1}\hat{\mu}\ell n\hat{\mu}$ with $\hat{\mu} \equiv \hat{g}^{-1}$. Then, for $t \leq \hat{t}_s$, $\eta(t)$ is reduced to $0(\hat{g}^{-1})$, that is, $|\eta(t)| = 0(\hat{g}^{-1})$

Proof: The proof is analogous to the proof for Theorem 3.1, details are omitted.

3.3. Regulation by High Gain State Feedback

In Section 3.1, we have derived conditions for the states x to have arbitrarily small sensitivities with respect to parameter variations and disturbances. We recognize that, in reality, very often we are only interested in the regulation of some of the state variables. In general, these variables can be expressed as linear combinations of the states which we have denoted by z in (3.3). We assume that there are ℓ of these variables to be regulated, that is, $z = D_o x$ is an ℓ-vector and $\ell < n$. Without loss of generality, it is assumed that rank $D_o = \ell$. In this section, we derive the conditions on D_o for the existence of a high gain state feedback control which regulates the variables z. Suppose that the feedback gain matrix K is designed such that condition (3.12), (3.13) are satisfied, we have the following theorem.

Theorem 3.4:

Let t_s be the time instant as defined in Theorem 3.1. Suppose that the system (3.9)-(3.11) when $f = 0$ is asymtotically stable, then the motion $z(t)$ is reduced to an $O(g^{-1})$ quantity, that is, $|z(t)| = O(g^{-1})$ for all $t \geq t_s$ if and only if

$$R(D_o^T) \subseteq R(K^T). \tag{3.51}$$

Proof: Sufficiency. Analogous to (2.18), x can be written as

$$x = Qx_1' + B_o(KB_o)^{-1}\omega \tag{3.52}$$

where ω is as in (3.14) and the columns of the $n\times(n-m)$ matrix Q span the null space of K, that is,

$$KQ = 0. \tag{3.53}$$

It follows that

$$z = D_oQx_1' + D_oB_o(KB_o)^{-1}\omega. \tag{3.54}$$

Theorem 3.1 gives $|\omega(t)| = O(g^{-1})$ for all $t \geq t_s$, hence $|z(t)| = O(g^{-1})$ if $D_oQ = 0$ which is equivalent to (3.51). Necessity is proved by contradiction. Suppose (3.51) is violated, then $D_oQ \neq 0$. From (3.54), since $|x_1'(t)| \neq O(g^{-1})$ for all $t \geq t_s$. The theorem is proved.

In the above theorem, it is shown that if the range of D_o^T lies in the range of K^T, then $z(t)$ can be reduced to zero as rapidly as desired since t_s is $O(g^{-1})$. Another interpretation of (3.51) is that the variables z are linear combinations of the fast variables ω. Since $\omega(t)$ decays

exponentially in the time scale gt, z(t) is reduced to zero in the same time scale. The following theorem states the condition for z(t) to be zeroed in a finite time interval using high gain state feedback.

Theorem 3.5:

Let t_f be a finite positive constant. Under the same assumption as in Theorem 3.4, the motion z(t) is reduced to an $0(g^{-1})$ quantity, that is, $|z(t)| = 0(g^{-1})$ for all $t \geq t_f$ if and only if

$$< Z_1^o + \delta Z_1 + \delta B_1 L | \mathcal{R}[M_1(I - \delta B(K(B_o + \delta B))^{-1}K)E_o] >^{\dagger} \subseteq \mathcal{N}(D_o Q) \qquad (3.55)$$

where M_1 is as in (2.21) and the notations in Corollary 3.1 are used.

Proof: From the proof of the previous theorem, we recognize that we only need to prove that (3.55) is necessary and sufficient for $|D_o Q x_1'(t)| = 0(g^{-1})$ for all $t \geq t_f$. Lemma C3 shows that there exists $t_f > 0$ such that

$$D_o Q x_1'(t) = D_o Q \int_0^t e^{[Z_1^o + \delta Z_1 + \delta B_1 L](t-\tau)} [E_1^o - \delta B_1 (K(B_o + \delta B))^{-1} K E_o] f(\tau) d\tau + 0\,(g^{-1}) \qquad (3.56)$$

for all $t \geq t_f$. In [33], it is shown that (3.55) is necessary and sufficient for the integral term in (3.56) to vanish. This proves the theorem.

---

†Let F be an $n \times n$ and G an $n \times m$ matrix, then $<F|\mathcal{R}(G)>$ denotes the subspace spanned by the columns of the matrices G, FG, ..., $F^{k-1}G$ where $k \leq n$.

Given the variables to be regulated, condition (3.55) characterizes the ranges of δA, δB and E_o which allow regulation to occur by applying high gain state feedback. Nevertheless, it is difficult to determine from (3.55) the ranges of δA, δB and E_o that satisfy it. The following corollary relates (3.51) with (3.55).

Corollary 3.2:

If the pair $(Z_1^o + \delta Z_1 + \delta B_1 L,\ E_1^o - \delta B_1[K(B_o + \delta B)]^{-1}KE_o)$ is controllable, then condition (3.55) becomes (3.51).

We note that (3.51) is a sufficient condition (3.55). When there is no information available about the ranges of δA, δB, then to guarantee regulation of $z(t)$, we have to take into account all possible parameter variations. In this case, we recognize from this corollary that condition (3.55) should be replaced by the condition (3.51). Thus (3.51) becomes both necessary and sufficient condition for regulation of $z(t)$. In view of Theorem 3.4, $z(t)$ can be reduced to zero as rapidly as desired. (3.51) can also be interpreted as a specification to the synthesis of matrix K. This aspect will be brought up again in Section 3.6.

3.4. High Gain Observer Feedback

We now begin to examine the behavior of the system (3.1), (3.2) when high gain feedback is applied through the high gain observer (3.6), (3.7), that is, the high gain feedback control is of the form $u = gK\hat{x}$ as it is given in (3.4). It is recognized that this closed loop system contains two groups of high gain loops. To facilitate our analysis, we express this system in terms of the state x of (3.1) and error e' between x and observer state $\hat{x}$,

$$\dot{x} = [A_o + \delta A + g(B_o + \delta B)K]x - g(B_o + \delta B)Ke' + E_o f \tag{3.57}$$

$$\dot{e}' = (\delta A + \hat{g}H\delta C + g\delta BK)x + (A_o + \hat{g}HC_o - g\delta BK)e' + E_o f \tag{3.58}$$

At this stage, we assume that the gains of the main loops and the observer loops are of the same magnitudes, that is, $g = \alpha\hat{g}$ when α is some positive scalar which is $O(1)$. The feedback gain matrix K of (3.4) is designed such that (3.12), (3.13) are satisfied and the matrix H of the high gain observer satisfies (3.42),(3.43). Under these assumptions, we first state the conditions for x(t) to have arbitrarily small sensitivities with respect to disturbances f. Then we establish conditions for arbitrarily small sensitivities with respect to parameter variations δA, δB and δC and disturbances f.

Theorem 3.6:

Let S be the matrix defined in (D5). Using the notations in Appendix D, let

$$\Sigma_{\delta_2} = \begin{bmatrix} B_2(K_2 + \tilde{K}_2) & -B_2\hat{K}_2 \\ \alpha^{-1}H_2(\tilde{C}_2 - \delta C_2) & \alpha^{-1}H_2C_2^o \end{bmatrix} \tag{3.59}$$

$$\Sigma_{\delta_1} = \begin{bmatrix} B_2(K_1 + \tilde{K}_1) & -B_2\hat{K}_1 \\ -\alpha^{-1}H_2(\tilde{C}_1 - \delta C_1) & \alpha^{-1}H_2C_1^o \end{bmatrix} \tag{3.60}$$

$$Z_\delta = \begin{bmatrix} A_{11} & 0 \\ F_{11} & \hat{A}_{11}^o \end{bmatrix} - \begin{bmatrix} A_{12} & 0 \\ F_{12} & \hat{A}_{12}^o \end{bmatrix} \Sigma_{\delta_2}^{-1}\Sigma_{\delta_1} . \tag{3.61}$$

Then the solution of system (3.57, (3.58) can be approximated by

$$\begin{bmatrix} x_1''(t) \\ \varepsilon_1(t) \end{bmatrix} = e^{Z_\delta t} \begin{bmatrix} x_1''(0) \\ \varepsilon_1(0) \end{bmatrix} + 0(g^{-1}) \tag{3.62}$$

and

$$\begin{bmatrix} x_2''(t) \\ \varepsilon_2(t) \end{bmatrix} = e^{g\sum_{\delta_2} t} \begin{bmatrix} x_2''(0) \\ \varepsilon_2(0) \end{bmatrix} + \sum_{\delta_2}^{-1}\sum_{\delta_1} \begin{bmatrix} x_1''(0) \\ \varepsilon_1(0) \end{bmatrix}$$

$$- \sum_{\delta_2}^{-1}\sum_{\delta_1} \begin{bmatrix} x_1''(t) \\ \varepsilon_1(t) \end{bmatrix} + 0(g^{-1}) \tag{3.63}$$

if and only if

$$\mathcal{R}[(I - S)E_o] \subseteq \mathcal{R}(H) \tag{3.64}$$

$$\mathcal{R}(E_o) \subseteq \mathcal{R}(B_o + \delta B) \tag{3.65}$$

and

$$\mathrm{Re}\lambda(\textstyle\sum_{\delta_2}) < 0. \tag{3.66}$$

Proof: As it is shown in Appendix D, system (3.57), (3.58) is equivalent to the system (D8) - (D11). Lemma C2 gives (3.63) and

$$\begin{bmatrix} x_1''(t) \\ \varepsilon_1(t) \end{bmatrix} = e^{Z_\delta t} \begin{bmatrix} x_1''(0) \\ \varepsilon_1(0) \end{bmatrix} + \int_0^t e^{Z_\delta(t-\tau)} \begin{bmatrix} E_1 \\ \tilde{E}_1 \end{bmatrix} f(\tau)d\tau + 0(g^{-1}). \tag{3.67}$$

From (D3),(D4),(D5) and (D14), $E_1 = 0$, $\tilde{E}_1 = 0$ providing that (3.64), (3.65) holds. This concludes the proof.

The following Corollary establishes the stability of system (3.57),(3.58).

Corollary 3.3:

For g sufficiently large, the system (3.57),(3.58) is asymptotically stable if and only if (3.64)-(3.66) and

$$\mathrm{Re}\lambda(Z_\delta) < 0 \tag{3.68}$$

holds. Furthermore, $\lambda(Z_\delta)$ are the transmission zeros of the triple

$$\left(\begin{bmatrix} A_o + \delta A & o \\ \delta A & A_o \end{bmatrix}, \begin{bmatrix} B_o + \delta B & o \\ \delta B & H \end{bmatrix}, \begin{bmatrix} K & -K \\ \delta C & C_o \end{bmatrix}\right)$$

and

$$\lambda(\Sigma_{\delta_2}) = \lambda\left(\begin{bmatrix} KB_o & -\alpha^{-1}KH \\ \delta CB + C_o\delta B & \alpha^{-1}C_oH \end{bmatrix}\right) . \tag{3.69}$$

Proof: $\lambda(Z_\delta)$ and $g\lambda(\Sigma_{\delta_2})$ are the $O(1)$ and $O(g)$ eigenvalues of the system (3.57),(3.58), in the limit when $g \to \infty$. The remainder of the corollary follows from the results in Sections 2.1 and 2.2. Details are omitted. From Theorem 3.6 and its corollary, we conclude that if E_o satisfies (3.64), (3.65), then the effects of disturbances f on the motion of system (3.57), (3.58) are attenuated by a factor of g^{-1}. The stability of this high gain feedback system depends on the parameter variations δA, δB and δC. When there are no parameter variations, that is, $\delta A = 0$, $\delta B = 0$ and $\delta C = 0$, the fast and slow eigenvalues as $g \to \infty$ are the eigenvalues of the matrices

$$g \begin{bmatrix} KB_o & -\alpha^{-1}KH \\ o & \alpha^{-1}C_oH \end{bmatrix} \tag{3.70}$$

and

$$Z^o = \begin{bmatrix} A_{11}^o - A_{12}^o K_s & A_{12}^o K_f^{-1}(K_f \hat{K}_s - \alpha^{-1}\hat{K}_f C_2^{o^{-1}} C_1^o) \\ \\ 0 & \hat{A}_{11}^o - \hat{A}_{12}^o C_2^{o^{-1}} C_1^o \end{bmatrix} \tag{3.71}$$

respectively where $\hat{K}_f$, $\hat{K}_s$ and as defined in (3.74) this can be shown by letting $\hat{M}_\delta = 0$ and $M_\delta = I_n$ in (D1) and assuming that the x-system has the form (3.9),(3.10). By the assumptions on K and H, (3.12), (3.13) (3.42), and (3.43), the eigenvalues of the matrices in (3.70), (3.71) have *negative real parts.* By continuity argument, $\lambda(Z_\delta)$ and $\lambda(\sum_{\delta_2})$ have negative real parts providing that the parameter variations $\|\delta A\|$, $\|\delta B\|$ and $\|\delta C\|$ are sufficiently small. Hence, we say that the stability of high gain observer feedback system is robust with respect to δA, δB and δC in the sense of (3.66) and (3.68) when the range of E_o satisfies (3.64), (3.65).

We now state the theorem regarding the conditions for x(t) to have arbitrarily small sensitivities with respect to parameter variations δA, δB, δC and disturbances f.

Theorem 3.7:

Let Z^o be the matrix defined in (3.71) and denote

$$\Sigma_2^o = \begin{bmatrix} (B_2^o + \delta B_2)K_f & -(B_2^o + \delta B_2)\hat{K}_f \\ \delta\hat{B}_2 K_f & \alpha^{-1}H_2 C_o^2 - \delta\hat{B}_2\hat{K}_f \end{bmatrix}, \tag{3.72}$$

$$\Sigma_1^o = \begin{bmatrix} (B_2^o + \delta B_2)K_f K_s & -(B_2^o + \delta B_2)K_f K_s \\ \delta\hat{B}_2 K_f K_s & \alpha^{-1}H_2 C_o^1 - \delta\hat{B}_2\hat{K}_f\hat{K}_s \end{bmatrix} \tag{3.73}$$

where

$$\hat{K}_s \equiv K_s\hat{M}^{-1}, \; K_f \equiv K_f\hat{M}^{-1}, \; \hat{\delta B} = \begin{bmatrix} \delta\hat{B}_1 \\ \delta\hat{B}_2 \end{bmatrix} \equiv \hat{M}\delta B. \tag{3.74}$$

Let the variable ϕ be defined

$$\phi = K_f K_s x_1' + K_f x_2' - K_f\hat{K}_s e_1 - \hat{K}_f e_2 \tag{3.75}$$

where x_1', x_2' and e_1, e_2 are the variables defined in (3.9), (3.10) and (3.39), (3.40), respectively. Then the solution of the high gain feedback system (3.57), (3.58) can be approximated by

$$\begin{bmatrix} x_1'(t) \\ e_1(t) \end{bmatrix} = e^{Z^o t} \begin{bmatrix} x_1'(0) \\ e_1(0) \end{bmatrix} + 0(g^{-1}) \tag{3.76}$$

$$\begin{bmatrix} x_2'(t) \\ e_2(t) \end{bmatrix} = e^{g\Sigma_2^o t} \begin{bmatrix} x_2'(0) \\ e_2(0) \end{bmatrix} + \Sigma_2^{o^{-1}} \Sigma_1^o \begin{matrix} x_1'(0) \\ e_1(0) \end{matrix}$$

$$- \Sigma_2^{o^{-1}} \Sigma_1^o \begin{bmatrix} x_1'(t) \\ e_1(t) \end{bmatrix} + 0(g^{-1}) \tag{3.77}$$

if and only if

$$R(E_o) \subseteq R(B_o),\ R(E_o) \subseteq R(H) \tag{3.78}$$

$$R(\delta A) \subseteq R(B_o),\ R(\delta A) \subseteq R(H) \tag{3.79}$$

$$R(\delta B) \subseteq R(B_o),\ R(\delta B) \subseteq R(H) \tag{3.80}$$

$$\delta C = 0 \tag{3.81}$$

and

$$\mathrm{Re}\lambda(\textstyle\sum_2^o) < 0. \tag{3.82}$$

Moreover, let $\sigma_{of} = \max \mathrm{Re}\lambda(\sum_2^o)$ and $t_{of} = |\sigma_{of}|^{-1} g^{-1} \ell n g^{-1}$. Then for all $t \geq t_{of}$, $\phi(t)$ is reduced to an $O(g^{-1})$ quantity, that is, $|\phi(t)| = O(g^{-1})$.

<u>Proof</u>: Sufficiency. If δB, δC satisfy conditions (3.79), (3.80), let $M_\delta = I_n$, $\hat{M}_\delta = 0$ in the transformation matrix M defined in (D1) and assume that the x-system (3.1) has the form (3.9), (3.10). Then by definition $x_1'' = x_1'$ $x_2'' = x_2'$, $\varepsilon_1 = e_1$, $\varepsilon_2 = e_2$ and $A_{ij} = A_{ij}^o + \delta A_{ij}$, $B_2 = B_2^o + \delta B_2$, $E_i = E_i^o$, $\hat{E}_i = \hat{E}_i^o$, $\tilde{K}_i$, $\tilde{C}_i$ and δC_i are null matrices and $K_1 = K_f K_s$, $K_2 = K_f$. Condition (3.78) implies δA_{ij} and δF_{ij} are null matrices and E_1^o, $\hat{E}_1^o$ are null if (3.77) holds. Lemma C2 gives (3.76), (3.77). Necessity is proved by contradiction. Details are omitted. This concludes the proof. We have the following corollary concerning stability.

<u>Corollary 3.4</u>:

Given that the parameter variations δA, δB, δC and disturbances f satisfying (3.78)-(3.82) then for g sufficiently large, the system (3.57), (3.58)

is asymptotically stable. Furthermore,

$$\lambda(\Sigma_2^o) = \lambda\left(\begin{bmatrix} KB_o & -\alpha^{-1}KH \\ C_o\delta B & \alpha^{-1}C_oH \end{bmatrix}\right) \tag{3.83}$$

and $\lambda(Z^o)$ are the transmission zeros of the triple

$$\left(\begin{bmatrix} A_o + \delta A & 0 \\ \delta A & A_o \end{bmatrix}, \begin{bmatrix} B_o + \delta B & 0 \\ \delta B & H \end{bmatrix}, \begin{bmatrix} K & -K \\ \delta C & C_o \end{bmatrix}\right).$$

Proof: By assumption K and H are designed such that (3.12), (3.13) and (3.42), (3.43) holds, hence $\mathrm{Re}\lambda(Z^o) < 0$. The proof is then analogous to that of Corollary 3.3, details are omitted.

In Theorem 3.6 and Theorem 3.7, it is shown that the sensitivity of $x(t)$ when high gain feedback control $u = gK\hat{x}$ is applied to the system depends on the ranges of the parameter variations matrices δA, δB, δC, the disturbance matrix E_o and the "input" matrix H in the high gain observer system (3.6), (3.7). If the ranges of δA, δB and E_o lie in the range of B_o and H simultaneously and there is no parameter uncertainty in the measurement matrix C, then the motion $x_1'(t)$ has $O(\mu)$ sensitivity with respect to parameter variations and disturbances and the influence of f and parameter variations to the motion $x_2'(t)$ for $t \geq t_{of}$ is also $O(\mu)$. The stability of the fast transients in $x_2'(t)$ depends on δB and is robust in the sense of (3.83). If the range conditions (3.78)-(3.80) do not hold and $\delta C \neq 0$, but conditions (3.64), (3.65) hold, then the sensitivity of $x_1'(t)$ with respect to f is $O(\mu)$. Furthermore, the stability of system

(3.57), (3.58) is only robust in the sense of (3.66) and (3.68). From (3.67), we see that if the range of E_o lies outside the range of H, then $\varepsilon_1(t)$ is sensitive to disturbance f and consequently $x_1'(t)$ is disturbed by f.

If the gains of the main loops and the observer loops are different in order of magnitudes, that is the products $\hat{g}g^{-1}$ or $g\hat{g}^{-1}$ tend to zero as g and $\hat{g}$ both tend to infinity, then instead of the two time scales, t and gt we have shown in Theorem 3.6, there are three time scales, t, gt and $\hat{g}t$. We now consider the case when $\lim_{g,\hat{g}\to\infty} g\hat{g}^{-1} = 0$, that is, roughly speaking, $\hat{g} >> g$.

Theorem 3.8:

Suppose that $\lim_{g,\hat{g}\to\infty} g\hat{g}^{-1} = 0$, then the solution of the system (3.57), (3.58) can be approximated by

$$\eta(t) = e^{\hat{g}C_oHt}\eta(0) + 0(\hat{g}^{-1}) \tag{3.84}$$

$$\begin{aligned} x_2'(t) = {} & e^{gK(B_o + \delta B)t}[x_2'(0) + K_f^{-1}(K_f\hat{K}_s - \hat{K}_fC_2^{o^{-1}}C_1^o)e_1(0) + K_sx_1'(0)] \\ & -K_f^{-1}(\hat{K}_f\hat{K}_s - \hat{K}_fC_2^{o^{-1}}C_1^o)e_1(t) - K_sx_1'(t) + 0(g^{-1}) \end{aligned} \tag{3.85}$$

$$\begin{bmatrix} e_1(t) \\ x_1'(t) \end{bmatrix} = e^{Z^ot}\begin{bmatrix} e_1(0) \\ x_1'(0) \end{bmatrix} + 0(g^{-1}) \tag{3.86}$$

if and only if the range conditions (3.77)-(3.80) and the condition (3.20) are satisfied.

Proof: Proof is analogous to that of Theorem 3.1, details are omitted. The three time scales property of the high gain feedback system (3.57), (3.58) when $\hat{g} >> g$ is evident from (3.84)-(3.86). The motions $\eta(t)$, $x_2'(t)$, $x_1'(t)$ and $e_1(t)$ can be characterized by eigenvalues of different orders of magnitudes, $\hat{g}\lambda(C_oH)$, $\hat{g}\lambda(K(B_o + \delta B))$ and $\lambda(Z^o)$. If the range conditions (3.77)-(3.80) are not satisfied but conditions (3.64), (3.65) hold, then analogous to Theorem 3.8, we have the following Theorem.

Theorem 3.9:

Using the same notations as in Appendix D, let

$$\Gamma_2 = [B_2(K_2 + \tilde{K}_2) - \hat{K}_2(C_oH)^{-1}H_2(\tilde{C}_2 - \delta C_2)] \tag{3.87}$$

$$\Gamma_1 = B_2[\hat{K}_1 - \hat{K}_2(C_oH)^{-1}H_2C_1^o \quad K_1 + \tilde{K}_1 - \tilde{K}_2(C_oH)^{-1}H_2(\tilde{C}_1 - \delta C_1)] \tag{3.88}$$

$$Z_1 = \begin{bmatrix} A_{11} & 0 \\ F_{11} + \hat{A}_{12}^o(C_oH)^{-1}H_2(\tilde{C}_1 - \delta C_1) & \hat{A}_{11}^o \end{bmatrix} -$$

$$\begin{bmatrix} A_{12} \\ F_{12} + \hat{A}_{12}^o(\tilde{C}_oH)^{-1}H_2(C_2 - \delta C_2) \end{bmatrix} \Gamma_2^{-1}\Gamma_1, \tag{3.89}$$

Then the sensitivities of the motions x(t) and e(t) with respect to f are of $O(g^{-1})$ if and only if conditions (3.64) and (3.65) hold. Furthermore, x(t) and e(t) can be characterized by the eignevalues $\hat{g}\lambda(C_oH)$, $g\lambda(\Gamma_2)$ and $\lambda(Z_1)$.

Proof: The proof is analogous to that of Theorem 3.8, details are omitted.

By comparing the solutions (3.76), (3.77) and (3.84)-(3.86), we recognize that for the class of parameter sanctions δA, δB and δC and disturbance matrix E_o satisfying conditions (3.78)-(3.81), the effect of having $\hat{g} >> g$ is that the fast eigenvalues, $\lambda\,(\sum_2^o)$ of (3.72), which are $O(g)$ decompose into two groups of eigenvalues of different magnitudes, $\hat{g}\lambda(C_oH)$ and $g\lambda(K(Bo + \delta B))$. Moreover, from Theorems 3.1 and 3.3, these eigenvalues, $g\lambda(C_oH)$ and $g\lambda(K(Bo + \delta B))$, are the "fast" eigenvalues of the high gain observer system (3.39)-(3.40) and the high gain state feedback system (3.9)-(3.11), respectively. Furthermore, the $O(1)$ eigenvalues of the high gain observer feedback system (3.57),(3.58) are the union of the sets of $O(1)$ eigenvalues $\lambda(A_{11}^o - A_{12}^o K_s)$ of (3.9)-(3.11) and $\lambda(\hat{A}_{11}^o - \hat{A}_{12}^o C_2^{o\;-1} C_1^o)$ of (3.39)-(3.41) which is evident from (3.71). Thus we conclude that the separation property of regular gain observer feedback systems holds if conditions (3.78)-(3.81) are satisfied and $\hat{g} >> g$. By comparing the results of Theorem 3.6 and 3.9, we recognize that this separation property does not hold even when the high gain observer feedback system (3.57), (3.50) is $O(\mu)$ sensitive to disturbance f.

We will not consider the case when $g >> \hat{g}$ since this implies the observer states are fed back even before the fast transients in the estimation error are over. Such considerations are in conflict with the concept of observer feedback.

3.5. Regulation by High Gain Observer Feedback

Under the assumption that K and H are designed such that (3.12), (3.13) and (3.42), (3.43), respectively, we now establish the conditions for the existence of a high gain observer feedback control that solves the partial states regulation problem described in Section 3.3, that is, the regulation of the variables $z = D_o x$. Before we state the theorem, it is essential to decompose x and e into slow and fast variables analogous to (3.52).

Lemma 3.1:

Assume that the matrix Σ_{δ_2} of (3.59) is nonsingular. Then the variables x and e can be expressed in terms of fast and slow variables as

$$\begin{bmatrix} x \\ e \end{bmatrix} = \Psi \begin{bmatrix} x_1'' \\ \varepsilon_1 \end{bmatrix} + \begin{bmatrix} B_o & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} KB_o & -\alpha^{-1}KH \\ \delta CB_o & \alpha^{-1}C_oH \end{bmatrix} \begin{bmatrix} \hat{\phi} \\ \hat{\eta} \end{bmatrix} \tag{3.90}$$

where $\hat{\phi}$ and $\hat{\eta}$ are the fast variables approximated by

$$\begin{bmatrix} \hat{\phi}(t) \\ \hat{\eta}(t) \end{bmatrix} = e^{gK_2\mathcal{B}_2 t} \begin{bmatrix} \hat{\phi}(0) \\ \hat{\eta}(0) \end{bmatrix} + O(g^{-1}) \tag{3.91}$$

and x_1'' and ε_1 are the slow variables approximated by

$$\begin{bmatrix} x_1''(t) \\ \varepsilon_1(t) \end{bmatrix} = e^{(A_1 - A_2K_2^{-1}K_1)t} \begin{bmatrix} x_1''(0) \\ \varepsilon_1(0) \end{bmatrix} + \int_0^t e^{(A_1 - A_2K_2^{-1}K_1)(t-\tau)} E_1 f(\tau)d\tau + O(g^{-1}) \tag{3.92}$$

and the matrices A_i, K_i, E_1 and B_2 are as defined in (D19), (D23)-(D25). In (3.90), the matrix Ψ is as defined in (D30), that is,

$$\begin{bmatrix} -K & K \\ \delta C & C_o \end{bmatrix} \Psi = 0_{(m+r) \times (2n-m-r)}. \tag{3.93}$$

Proof: See Appendix D.

The theorem on regulation of z(t) is as follows.

Theorem 3.10:

Let $\hat{M}_1^T$ be the matrix whose columns span the null space of H^T, that is,

$$\hat{M}_1 H = 0. \tag{3.94}$$

Let Ψ_1, Ψ_2 be the $n \times (2n-m-r)$ submatrices of Ψ,

$$\Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \tag{3.95}$$

$M_{\delta_1}^T$ be the matrix whose columns span the null space of $(B_o + \delta B)^T$, that is,

$$M_{\delta_1}(B_o + \delta B) = 0 \tag{3.96}$$

Let $\hat{t}_r$ be a finite positive real number and matrix S be defined as in (D5).

Suppose that high gain observer feedback control (3.4) is applied to system (3.1). Then there exists $\hat{t}_r$ such that for all $\hat{t} \geq t_r$,

the regulated variables $z(t) = D_o x(t)$ are reduced to $0(g^{-1})$, that is $|z(t)| = 0(g^{-1})$ if and only if system (3.57), (3.58) is asymptotically stable and

$$\langle A_1 - A_2K_2^{-1}K_1 | R(\begin{bmatrix} M_{\delta_1} \\ \hat{M}_1(I - S) \end{bmatrix} E_o) \rangle \subseteq N(D_o\Psi_1). \tag{3.97}$$

Proof: Lemma 3.1 establishes the decomposition of the variables x and e into fast and slow variables. Then the proof follows closely that of Theorem 3.5, details are omitted.

Corollary 3.5:

$$\text{Let } \begin{bmatrix} M_{\delta_1} \\ \hat{M}_1(I - S) \end{bmatrix} E_o = E_r. \tag{3.98}$$

If the pair $(A_1 - A_2 K_2^{-1}K_1, E_r)$ is controllable then condition (3.97) is

$$R\left(\begin{bmatrix} D_o^T \\ 0 \end{bmatrix}\right) \subseteq R\left(\begin{bmatrix} K^T & \delta C^T \\ -K^T & C_o^T \end{bmatrix}\right) \tag{3.99}$$

Proof: If the pair $(A_1 - A_2K_2^{-1}K_1, E_r)$ is controllable, then $\langle A_1 - A_2K_2^{-1}K_1, E_r\rangle = R^{2n-m-r}$. (3.97) holds in this case if and only if $D_o\Psi_1 = 0$ or equivalently $[D_o \quad 0]\Psi = 0$. Since columns of Ψ span $N([\begin{smallmatrix} K & -K \\ \delta C & Co \end{smallmatrix}])$, (3.99) is obtained.

We note from their definitions that the matrices A_1, A_2, K_1, K_2 depend on all parameter variations δA, δB and δC while the matrices M_{δ_1}, S depend only on δB. Hence in order to guarantee regulation of z(t) for all possible δA, δB and δC, condition (3.97) in Theorem 3.10 is replaced by

(3.99). The following lemma explores the implications of condition (3.99) on the structure of matrix D_o.

Lemma 3.2:

A necessary condition for (3.99) to hold is that

$$R(D_o^T) \subseteq R([C_o^T + \delta C^T])$$

Proof: (3.99) implies the existence of a matrix $P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ such that

$$\begin{bmatrix} D_o^T \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & C_o^T + \delta C^T \\ -K^T & C_o^T \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} . \qquad (3.100)$$

This lemma shows that when no restriction is imposed on parameter variations, then regulation of z(t) can be maintained only if the regulated variables z appears as linear combinations of the measured variables y. We note that so far we assume that high gain feedback is made of an observer whose sensitivity with respect to parameter variations and disturbances is not reduced to $O(\hat{g}^{-1})$ by the observer high gain loops. By assuming that the disturbances and parameter variations are "rejected" by the high gain observer, we then have the following conditions on regulation of z(t).

Theorem 3.11:

Suppose that the matrices δA, δB, δC and E_o satisfy (3.47)-(3.50) and the high gain observer has the properties of Theorem 3.3. Let the condition (3.82) be satisfied and $\hat{t}_r$ be a finite positive constant. Then there exists $\hat{t}_r$ such that for all $t \geq \hat{t}_r$, z(t) is reduced to an $O(g^{-1})$ quantity, that is, $|z(t)| = O(g^{-1})$ if and only if

$$\langle A_{11} - A_{12}K_2^{-1}K_1 | \mathcal{R}(M_{\delta_1} E_o)\rangle \subseteq \mathcal{N}(D_o Q) \tag{3.101}$$

and

$$\mathrm{Re}\lambda(A_{11} - A_{12}K_2^{-1}K_1) < 0 \tag{3.102}$$

where Q is the matrix in (3.53), M_{δ_1} as in (3.96) and A_{ij}, K_i are as defined in (D12)-(D16).

Proof: If δA, δB, δC and E_o satisfy conditions (3.47)-(3.50) let $\hat{M}_\delta = 0$ in the transformation matrix M of (D1). Then $\varepsilon_1 = e_1$ and $\varepsilon_2 = e_2$ and $\tilde{K}_i$, $\check{C}_i$, δC_i and δF_{ij} are null matrices. Since $\delta C = 0$, we can take Ψ in (3.93) to be

$$\Psi = \begin{bmatrix} Q & N \\ 0 & N \end{bmatrix} \tag{3.103}$$

where N is as in (3.19), that is, it satisfied $C_o N = 0$. Then (3.90) becomes

$$x = Qx_1'' + Ne_1 + B_o(KB_o)^{-1}\hat{\phi} + KH(CoH)^{-1}\hat{\eta} . \tag{3.104}$$

If (3.82) holds, then

$$x_1''(t) = e^{(A_{11} - A_{12}K_2^{-1}K_1)t} x_1''(0) + A_{12}(K_2^{-1}\hat{K}_2 C_2^{o^{-1}} C_1^o - \hat{K}_2^{-1}K_1)e_1(t)$$

$$+ \int_0^t e^{(A_{11} - A_{12}K_2^{-1}K_1)(t-\tau)} E_1 f(\tau)d\tau + 0(g^{-1}) \tag{3.105}$$

$$e_1(t) = e^{(\hat{A}_{11}^o - \hat{A}_{12}^o C_2^{o^{-1}} C_1^o)t} e_1(0) + 0(g^{-1}) \tag{3.106}$$

and $\hat{\phi}(t)$, $\hat{\eta}(t)$ are given in (D26). The remaining of the proof is analogous to that of Theorem 3.5.

Corollary 3.6:

If the pair $(A_{11} - A_{12}K_2^{-1}K_1, M_{\delta_1}E_o)$ is controllable, then (3.101) is replaced by

$$R(D_o^T) \subseteq R(K^T) \tag{3.107}$$

which is the same condition as (3.51).

Condition (3.107) implies that $\ell \leq m$, that is, the number of regulated variables should not be more than the number of control variables. The coincidence of condition (3.107) and (3.51) agrees with intuition. Since the high gain observer rejects the parameter variations and disturbances and thus provides estimated states $\hat{x}$ which tracks x asymptotically, the condition for regulation of z(t) should be the same whether high gain state feedback or high gain observer feedback is employed. However, we note that if high gain state feedback is used, the transient responses are faster, that is, z(t) is reduced to $O(g^{-1})$ after the time t_s which is $O(g^{-1})$. Since the matrices A_{ij}, K_i and M_{δ_1} depends on parameter variations δA, δB, in order to guarantee regulation of z(t) for all possible parameter variations, we will focus on condition (3.107) when we consider the synthesis of matrix K in the next section.

If $C_o = D_o$, that is, the measured variables are also the regulated variables, and $m = r = \ell$, then the following corollary of Theorem 3.11 is obtained.

Corollary 3.7:

Let $C_o = D_o$ and $m = r = \ell$. Suppose it is required to regulate z(t) for all possible parameter variations δA, δB and disturbance matrix

E_o, then under the same assumptions of Theorem 3.11, the conditions (3.101), (3.102) are replaced by

$$\mathcal{R}(C_o^T) = \mathcal{R}(K^T) \tag{3.108}$$

and the condition that the transmission zeros of the triple $(C_o, A_o + \delta A, B_o + \delta B)$ all lie in the open left half plane.

Proof: If $C_o = D_o$, $m = r = \ell$, then (3.107) is (3.108). Consequently, $\lambda(A_{11} - A_{12}K_2^{-1}K_1)$ are the transmission zeros of the triple $(C_o, A_o + A, B_o + \delta B)$ from the results of Section 2.2.

We see that when the number of control variables is the same as the number of measured variables and the measured variables are the variables to be regulated, then in spite of the use of a "disturbance rejecting" high gain observer in the high gain feedback loops, the finite eigenvalues of system (3.51), (3.58) are determined by the system parameters $A_o + \delta A$, $B_o + \delta B$ and C_o. In fact, Theorem 2.2 shows that these finite eigenvalues coincide with the finite eigenvalues of the system (3.1) when high gain output feedback $u = gC_o x$ is applied. If $m > \ell$, we recognize that there exists some degrees of freedom which allows the placement of the finite eigenvalues. These aspects of the synthesis of the matrices K and H are the subject of the next section.

3.6 Synthesis Problems

So far in this chapter, we have demonstrated that high gain feedback is capable of reducing the sensitivities of control systems with respect to parameter variations and disturbances to as desired providing that the ranges of δA, δB and E_o lie in the proper ranges. The range

conditions that involve the range of B_o and C_o are dictated by the input output structures of the open loop system. The ones that involve range of K and H can be manipulated since matrices K and H are design parameters. However, we recall that these matrices also determine the dynamical characteristics of the high gain feedback system. Hence, we are faced with synthesis problems with two objectives, that is the matrices K and H are designed such that sensitivity reduction by high gain feedback is achievable while the high gain feedback system possesses desirable characteristics. We recognize that in multivariable systems, the solution to some feedback gains design problems are nonunique and our success in achieving the design goal depends on the degrees of freedom available to satisfy both objectives.

The range conditions that we want to manipulate are as follows

$$\mathcal{R}(\delta A) \subseteq \mathcal{R}(H), \ \mathcal{R}(\delta B) \subseteq \mathcal{R}(H), \mathcal{R}(E_o) \subseteq \mathcal{R}(H) \qquad (3.47)\text{-}(3.49)$$

$$\mathcal{R}(D_o^T) \subseteq \mathcal{R}(K^T). \qquad (3.107)$$

For design purposes, there is no loss of generality to assume that $\mathcal{R}(\delta A) \subseteq \mathcal{R}(E)$, $\mathcal{R}(\delta B) \subseteq \mathcal{R}(E)$ and $\mathcal{R}(E_o) \subseteq \mathcal{R}(E)$. Thus, in place of (3.47)-(3.49), we consider the condition

$$\mathcal{R}(E) \subseteq \mathcal{R}(H). \qquad (3.109)$$

The design problems regarding matrices K and H can be defined formally as Problem K and Problem H.

Problem K:

Design matrix K such that

(i) $\mathcal{R}(D_o^T) \subseteq \mathcal{R}(K^T)$ (3.107)

(ii) $\mathrm{Re}\lambda(A_{11}^o - A_{12}^o K_s) < 0$ (3.12)

(iii) $\mathrm{Re}\lambda(KB_o) < 0.$ (3.13)

Problem H:

Design matrix H such that

(i) $\mathcal{R}(E) \subseteq \mathcal{R}(H)$ (3.109)

(ii) $\mathrm{Re}\lambda(\hat{A}_{11}^o - \hat{A}_{12}^o C_2^{o^{-1}} C_1^o) < 0$ (3.42)

(iii) $\mathrm{Re}\lambda(C_o H) < 0.$ (3.43)

Although the specifications (ii) and (iii) demand only asymptotic stability of the fast and slow modes of the high gain feedback systems, feasibilities for slow and fast pole placement will also be investigated. We have not taken into account the effects of parameter variations on stability in these specifications since these are not the problems of concern for the design of K and H. We recognize that, from duality, the design procedures to solve problems K and H are the same. Therefore, we derive solvability conditions only for Problem K. First, we state the constraints of requirements (ii) and (iii) on the structure of matrix K.

Lemma 3.3:

The matrix KB_o is nonsingular if and only if

$$\mathcal{R}(B_o) \cap \mathcal{N}(K) = \emptyset. \tag{3.110}$$

Proof: Necessity. Suppose $(KB_o)^{-1}$ exists and there exists a vector q such that $\mathcal{R}(B_o q) \subset \mathcal{N}(K)$. But this implies rank $(KB_o) < m$ which is a contradiction. To prove sufficiency, (3.110) implies there exists no nonzero vector q such that $KB_o q = 0$. It follows that KB_o is nonsingular.

Corollary 3.8:

Condition (3.110) is equivalent to

$$\mathcal{R}(K^T) \cap \mathcal{N}(B_o{}^T) = \emptyset. \tag{3.111}$$

Lemma 3.3 and its Corollary shows that (iii) restricts the range of K^T to be outside the null space of B_o^T. This observation leads to the first solvability condition of Problem K.

Lemma 3.4: A necessary condition for the solution of Problem K to exist is that

$$\mathcal{R}(D_o^T) \cap \mathcal{N}(B_o{}^T) = \emptyset \tag{3.112}$$

Proof: (i) requires $\mathcal{R}(D_o^T) \subseteq \mathcal{R}(K^T)$ but (iii) restricts $\mathcal{R}(K^T) \cap \mathcal{N}(B_o^T) = \emptyset$, hence (3.112) is necessary to satisfy both (i) and (iii).

Thus, we assume that matrix D_o satisfies (3.112). This assumption allows us to introduce the transformation $x' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = M'x$ where

$$M' = \begin{bmatrix} M_1 \\ \\ M_2' \end{bmatrix} \tag{3.113}$$

M_1 is as defined in (2.10) and the $m \times n$ matrix M_2' is chosen such that

$$\mathcal{R}(D_o^T) \subseteq \mathcal{R}(M_2'^T). \tag{3.114}$$

<u>Lemma 3.5</u>:

Let

$$KM'^{-1} = [K_f'K_s' \quad K_f'] \tag{3.115}$$

and
$$\omega' = KM'^{-1}x'. \tag{3.116}$$

Then the solution of the system

$$\dot{x} = (A_o + gB_oK)x \tag{3.117}$$

can be approximated by

$$x_1'(t) = e^{(A_{11}'^{o} - A_{12}'^{o}K_s')t}\, x_1'(0) + 0(g^{-1}) \tag{3.118}$$

$$\omega'(t) = e^{gK_f'B_2^o}\omega'(0) + 0(g^{-1}). \tag{3.119}$$

where

$$M'A_oM'^{-1} = \begin{bmatrix} A_{11}'^{o} & A_{12}'^{o} \\ \\ A_{21}'^{o} & A_{12}'^{o} \end{bmatrix} \tag{3.120}$$

and B_2^o is as defined in (2.9).

Proof: The proof is analogous to that of Theorem 2.1, details are omitted.

Clearly, (ii) and (iii) are equivalent to

(ii') $\mathrm{Re}\lambda(A'^{o}_{11} - A'^{o}_{12}K'_{s}) < 0$

(iii') $\mathrm{Re}\lambda(K'_{f}B^{o}_{2}) < 0$

respectively. The motivation behind the use of the transformation M' is clear from the following lemma.

Lemma 3.6:

Let Γ be an $\ell \times m$ matrix satisfying

$$D_o^T = M_2'^{T}\,\Gamma^T \tag{3.121}$$

then

$$D_o M'^{-1} = [0_{\ell x(n-m)} \quad \Gamma] \tag{3.122}$$

Proof: Direct computation gives (3.122).

We thus have the following solvability condition for Problem K.

Theorem 3.12:

Let Π be an $m \times (m-\ell)$ matrix whose columns span the null space of Γ defined in (3.121), that is,

$$\Gamma\Pi = 0 \tag{3.123}$$

Suppose that D_o satisfies (3.112), then a necessary and sufficient condition for the existence of a matrix K which fulfills the condition (3.107), (ii') and (iii') is that the pair $(A'^o_{11}, A'^o_{12}\Pi)$ be stabilizable.

Proof: Condition (3.107) holds if and only if there exists an $m \times \ell$ matrix P such that

$$K'^T_s K'^T_f P = 0 \tag{3.124}$$

and

$$K'^T_f P = \Gamma^T. \tag{3.125}$$

(iii') means that K'^{-1}_f exists, hence (3.124) becomes

$$\Gamma K'_s = 0. \tag{3.126}$$

Thus, (3.107) is equivalent to the matrix K'_s has the form

$$K'_s = \Pi K''_s \tag{3.127}$$

where K''_s is a matrix of appropriate dimensions. Substituting (3.127) into (ii'), it becomes

$$\mathrm{Re}\lambda(A'^o_{11} - A'^o_{12}\Pi K''_s) < 0. \tag{3.128}$$

Clearly, there exists matrix K''_s such that (3.128) is satisfied if and only if the pair $(A'^o_{11}, A'^o_{12}\Pi)$ is stabilizable. Since (iii') implies B^o_2 is nonsingular, there exists K'_f such that (iii') is fulfilled. This concludes the proof of this theorem.

The condition for the existence of matrix K that places the eigenvalues

subjected to the structural constraint (3.107) is evident from the proof. We have the following theorem.

Theorem 3.13K:

Let p_i, $i = 1,\ldots,n-m$ and gq_j, $j = 1,\ldots,m$ be the desired locations for the slow and fast eigenvalues of the system (3.117). Then under the same assumptions of Theorem 3.12, the necessary and sufficient condition for the existence of a matrix K such that condition (3.107) is fulfilled and

$$\lambda(A'^{o}_{11} - A'^{o}_{12}K'_s) = p_i, \tag{3.129}$$

and

$$\lambda(K'_f B^o_2) = q_j \tag{3.130}$$

is that the pair $(A'^{o}_{11}, A'^{o}_{12}\Pi)$ be controllable.

If $\ell = m$, then clearly $\Gamma = I_m$ and $\Pi = 0$. The following corollary is obtained.

Corollary 3.9K:

If $\ell = m$, then the necessary and sufficient condition in Theorem 3.12 becomes

$$\mathrm{Re}\lambda(A'^{o}_{11}) < 0 \tag{3.131}$$

that is, the transmission zeros of the triple (D_o, A_o, B_o) lie in the open left half complex plane.

Proof: If $\ell = m$, then $\Gamma = I_m$ and $\Pi = 0$ and (3.128) becomes (3.131). From the results of Section 2.2, $\lambda(A'^{o}_{11})$ are the transmission zeros of the triple (D_o, A_o, B_o).

In Theorem 3.12 and 3.13K, the solvability of Problem K is stated in terms of Π which depends on the transformation matrix M_2'. We now show that the controllability of the pair $(A'^{o}_{11}, A'^{o}_{12}\Pi)$ is dependent only on system matrices B_o and D_o.

Theorem 3.14K:

Let Ω be an $n \times (m-\ell)$ matrix with rank $\Omega = m-\ell$ such that

$$\mathcal{R}(\Omega) \cap [\mathcal{R}(D_o^T) \cup \mathcal{N}(B_o^T)] = \emptyset \tag{3.132}$$

Then the controllability of the pair $(A'^{o}_{11}, A'^{o}_{12}\Pi)$ is equivalent to the controllability of the pair $(M_1A_oS_1, M_1A_o\Omega)$ where the inverse of M' of (3.113) is defined

$$M'^{-1} = [S_1 \ \underbrace{S_2}_{m}] \tag{3.133}$$

Proof: By definition (3.120)

$$A'^{o}_{11} = M_1A_oS_1 \quad , \; A'^{o}_{12} = M_1A_oS_2.$$

To prove the equivalence statement, we need only to show $\mathcal{R}(M_1A_oS_2\Pi) = \mathcal{R}(M_1A_o\Omega)$. Let M_2' be chosen such that

$$M_2' = \begin{bmatrix} D_o \\ G \end{bmatrix} \tag{3.134}$$

where G is a $(m-\ell) \times n$ matrix such that rank $M_2' = m$.

Denote

$$S_2 = [\underbrace{S_2^1}_{\ell} \quad \underbrace{S_2^2}_{m-\ell}] \tag{3.135}$$

Since $M_2'S_2 = I_m$,

$$D_o S_2^2 = 0_{\ell \times (m-\ell)} \tag{3.136}$$

and

$$GS_2^2 = I_{m-\ell}. \tag{3.137}$$

For the choice of M_2' in (3.134),

$$S_2\Pi = S_2^2 \tag{3.138}$$

It follows that

$$\mathcal{R}(S_2\Pi) \subset \mathcal{N}(D_o) \tag{3.139}$$

and

$$\mathcal{R}(S_2\Pi) \cap \mathcal{N}(G) = \emptyset \tag{3.140}$$

Since $\mathcal{R}(G^T)$, $\mathcal{R}(D_o^T)$ and $\mathcal{N}(B_o^T)$ span R^n, (3.139), (3.140) leads to

$$\mathcal{R}(S_2\Pi) \cap [\mathcal{R}(D_o^T) \cup \mathcal{N}(B_o^T)] = \emptyset \tag{3.141}$$

This concludes the proof.

We now summarize the solvability of Problem H in the following Theorem.

Theorem 3.13H:

Assume that

$$R(E) \cap N(C_o) = \emptyset. \tag{3.142}$$

Let $\bar{M}$ be as $n \times n$ matrix defined by

$$\bar{M}C_o^T = \begin{bmatrix} 0 \\ C_2 \end{bmatrix}, \qquad \bar{M} = \begin{bmatrix} \bar{M}_1 \\ \bar{M}_2 \end{bmatrix} \tag{3.143}$$

where $\bar{M}_2$ is an $m \times n$ matrix

and

$$\bar{M}A_o^T\bar{M}^{-1} = \begin{bmatrix} \bar{A}_{11}^o & \bar{A}_{12}^o \\ \bar{A}_{21}^o & \bar{A}_{22}^o \end{bmatrix}. \tag{3.144}$$

Let $\bar{\Gamma}$ be an $r \times p$ matrix satisfying

$$E = \bar{M}_2 \bar{\Gamma}^T \tag{3.145}$$

and let $\bar{\Gamma}$ be an $r \times (r-p)$ matrix whose columns span the null space of $\bar{\Gamma}$, that is,

$$\bar{\Gamma}\bar{\Pi} = 0 \tag{3.146}$$

Then a necessary and sufficient condition for the existence of a matrix H such that (3.109) is satisfied and $\lambda(\hat{A}_{11}^o - \hat{A}_{12}^o C_2^{o^{-1}} C_1^o)$ and $\lambda(C_oH)$ are positioned at desired locations is that the pair $(\bar{A}_{11}^o, \bar{A}_{12}^o\bar{\Pi})$ be controllable.

Proof: By letting $K \leftrightarrow H^T$, $B_o \leftrightarrow C_o^T$, $D_o^T \leftrightarrow E_o$, $A_o \leftrightarrow A_o^T$ and applying Theorem 3.13K, this theorem is proved.

The dual theorem of Theorem 3.14K is stated below.

Theorem 3.14H:

Let $\overline{\Omega}$ be an $n \times (r-p)$ matrix with rank $\overline{\Omega} = r-p$ such that

$$R(\overline{\Omega}) \cap [R(E_o) \cup N(C_o)] = \emptyset. \tag{3.147}$$

Then the controllability of the pair $(\overline{A}_{11}^{o}, \overline{A}_{12}^{o}\overline{\Pi})$ is equivalent to the controllability of the pair $(\overline{M}_1 A_o^T \overline{S}_1, \overline{M}_1 A_o^T \overline{\Omega})$ where the inverse of $\overline{M}$ of (3.143) is defined

$$\overline{M}^{-1} = [\overline{S}_1 \quad \underbrace{\overline{S}_2}_{r}]. \tag{3.148}$$

If $p = r$, then $\overline{\Gamma} = I_r$ and $\overline{\Pi} = 0$. The dual corollary of Corollary 3.9K is as follows.

Corollary 3.9H:

If $p = r$, then the necessary and sufficient condition in Theorem 3.13H becomes

$$\mathrm{Re}\lambda(\overline{A}_{11}^{o}) < 0 \tag{3.149}$$

that is, the transmission zeros of the triple (C_o, A_o, E) all lie in the open left half complex plane.

CHAPTER 4

VARIABLE STRUCTURE FEEDBACK SYSTEMS WITH SLIDING MODE

Another class of feedback controls capable of reducing parameter sensitivities and rejecting disturbances is the so-called variable structure control. Basically, it is a feedback control discontinuous on a switching surface $s(x) = 0$,

$$u_i(x) = \begin{cases} u_i^+(x) & s_i(x) > 0 \\ u_i^-(x) & s_i(x) < 0 \end{cases} \tag{4.1}$$

where u_i and s_i denote the i^{th} component of the m-vectors u and s, respectively. The class of control systems with discontinuous feedback control (4.1) is called variable structure systems (VSS) and has been developed in the USSR in the last fifteen years [14,15,16,17,18]. It has found applications in control of a wide range of processes in steel, power, chemical and aerospace industries. For a recent survey, see [19]. The salient feature of VSS is that the so-called sliding mode occurs on the switching surface $s(x) = 0$. While in sliding mode, the feedback system may be insensitive to parameter variations and disturbances.

The organization of this chapter is as follows. We first review the properties of sliding mode in VSS. Then we establish the relationship between high gain feedback systems and VSS with sliding mode. We reveal the insensitivity property of sliding mode by drawing an analogy with high gain feedback systems which we have investigated in the previous chapters. First, the case when all the states are accessible

is considered. We then examine the behavior of VSS with Luenburger observer feedback. A variable structure observer which is closely related to the high gain observer of Section 3.2 is developed. Finally, we look into VSS with variable structure observer feedback.

4.1. Properties of Sliding Mode

To illustrate some of the fundamental concepts in the theory of variable structure systems with sliding mode, we consider a second order linear time-invariant system,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= ax_2 - bu, \qquad a, b > 0 \end{aligned} \tag{4.2}$$

let the variable structure control law be

$$u = \begin{cases} \alpha x_1 \quad , & x_1 s > 0 \\ -\alpha x_1 \quad , & x_1 s < 0 \end{cases} \tag{4.3}$$

where the switching line is

$$s = cx_1 + x_2 = 0, \quad c > 0. \tag{4.4}$$

The phase plane trajectories of the closed loop system (4.2),(4.3) are illustrated in Figure 4.1. They are a combination of phase trajectories of two linear feedback systems with $u = \alpha x_1$ and $u = -\alpha x_1$. One of these systems is marginally stable and the other is unstable. By selecting α such that s and $\dot{s}$ have opposite signs in the neighborhood of $s = 0$, that is,

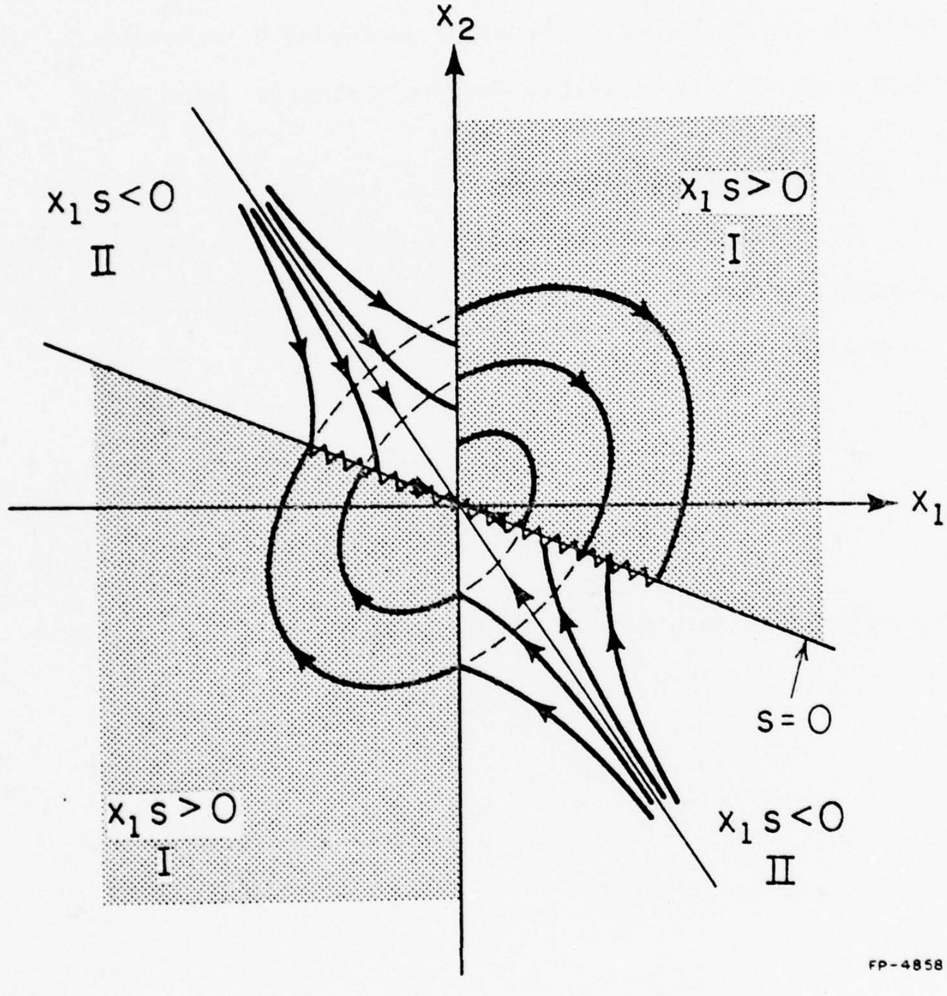


Figure 4.1 Sliding mode in a second order VSS.

$$\lim_{s \to -0} \dot{s} > 0 \text{ and } \lim_{s \to +0} \dot{s} < 0, \tag{4.5}$$

the phase trajectories will continue along the switching line (4.4). We call the motion along the switching line sliding mode. We note that sliding mode is not a part of trajectories which belong to either of the two feedback systems. When system (4.2), (4.3) is in sliding mode, its trajectories satisfy $s = cx_1(t) + x_2(t) = 0$, that is, $x_1(t)$ is described by the first order system

$$\dot{x}_1 = - cx_1. \tag{4.6}$$

We note that there is a reduction in system order when sliding mode occurs. The reduced order system is obtained by the equivalent control method proposed by Utkin [38]. According to his method, u is solved from the algebraic equation

$$\dot{s} = cx_2 + ax_2 - bu = 0 \tag{4.7}$$

The unique solution of (4.7) called the equivalent control u_{eq} is then substituted for u in (4.2) and (4.6) is obtained by restricting x_1, x_2 to satisfy $s = 0$. We call (4.6) the equivalent control system. It is dependent only on the parameter c of the switching plane $s = 0$ of (4.4). This insensitivity property in sliding mode depends on the system structure (4.2). In general, the equivalent control system which describes the dynamical behavior of the VSS in sliding mode depends both on system parameters and switching plane parameters. The design of VSS with sliding mode thus involves the synthesis of the switching planes

such that the equivalent control system is asymptotically stable. In this example, by choosing c to be positive guarantees the asymptotic stability of (4.6) and in fact $-c$ is the eigenvalue of (4.6).

Another important consideration in the design of VSS in the condition for the system trajectory to reach the switching planes. For the linear system (4.2), it happens that by choosing the variable structure control as in (4.3), all its trajectories reach the switching line. For illustration purposes, we consider the system (4.2) subjected to parameter variations and disturbance, that is,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= ax_2 - bu + w(t) \end{aligned} \tag{4.8}$$

where $w(t)$ represents an unknown disturbance and $|w(t)| < m_1$. The parameter variations in a and b are such that $|a| < m_2$ and $|b| < m_3$. The only information we have regarding a, b and $w(t)$ is their bounds m_1, m_2 and m_3. We also assume that the sign of b is known and disturbance $w(t)$ is accessible. If we adopt the variable structure control as in (4.3), the system trajectories may not move towards the switching plane (4.4) due to the presence of $w(t)$. A sufficient condition for the trajectories to move towards $s = 0$ is that

$$\dot{s}s < 0. \tag{4.9}$$

If we let the variable structure control be

$$u = \Psi_2 x_2 + \Psi_w w \tag{4.10}$$

where

$$\Psi_2 = \begin{cases} \alpha_2 \mathrm{sgn}(b), & x_2 s > 0 \\ \\ -\alpha_2 \mathrm{sgn}(b), & x_2 s < 0, \end{cases} \tag{4.11}$$

and $\alpha_2 > 0$,

$$\Psi_w = \begin{cases} \alpha_w \mathrm{sgn}(b), & ws > 0 \\ \\ -\alpha_w \mathrm{sgn}(b), & ws < 0 \end{cases} \tag{4.12}$$

and $\alpha_w > 0$, then condition (4.9) is satisfied if

$$\alpha_2 > \frac{c + m_2}{m_3}, \qquad \alpha_w > \frac{1}{m_3} . \tag{4.13}$$

We note that the equivalent control system for the VSS (4.8), (4.10) is (4.6) which is the same for the VSS (4.2), (4.3). This clearly exhibits the insensitivity with respect to parameter variations and the disturbance rejection capability of VSS with sliding mode.

If the disturbance w(t) is not measurable, then by letting the variable structure control be

$$u = \Psi_2 x_2 + \Psi_r \tag{4.14}$$

where Ψ_2 is as in (4.11) and

$$\Psi_r = \begin{cases} \alpha_r \mathrm{sgn}(b), & s > 0 \\ \\ -\alpha_r \mathrm{sgn}(b), & s < 0 \end{cases} \tag{4.15}$$

and $\alpha_r > 0$, the condition (4.9) is satisfied if

$$\alpha_2 > \frac{c + m_2}{m_3}, \quad \alpha_3 > \frac{m_1}{m_3} \quad . \tag{4.16}$$

The difference between the variable structure controls (4.10) and (4.14) is the component associated with $w(t)$. When the disturbance $w(t)$ is accessible, then it is 'fed forward" with a switching gain as in (4.10). If $w(t)$ cannot be measured, then a relay function is adopted as in (4.14). This determination of the variable structure control law according to the accessibility of the disturbances carries over in the multivariable case.

We now summarize some of the results in [19,37,38] which we need later. Consider the linear multivariable system

$$\dot{x} = Ax + Bu + E_o f \tag{4.17}$$

where as in (3.1), the state $x \in R^n$, the control $u \in R^m$ and the disturbance $f \in R^P$. Assuming that f is not measurable, we let the variable structure control law be of the form

$$u(x) = \Psi^o x + \psi_b \tag{4.18}$$

where

$$\Psi^o = \Psi \mathrm{diag}(\mathrm{sgn}\, x_1, \ldots, \mathrm{sgn}\, x_n) \tag{4.19}$$

$$\Psi^T = (\psi_1, \ldots, \psi_m) \tag{4.20}$$

$$\psi_i = \begin{cases} r_i & , s_i(x) > 0 \\ -r_i & , s_i(x) < 0 \end{cases} , \; i = 1, \ldots, m, b \tag{4.21}$$

and $s_i(x) = 0$ are the switching planes. We define an m-vector $s(x)$ whose i^{th} component is denoted as $s_i(x)$ by

$$s(x) = [s_1(x),\ldots,s_m(x)]^T = Kx = 0. \tag{4.22}$$

In (4.20), (4.21), ψ_i and r_i, $i = 1,\ldots,m$, are n-vectors and ψ_b and r_b are m-vectors. We note that u_i, the i^{th} component of u, is discontinuous on $s_i(x) = 0$ and is a linear function in x if x is not on any of the switching planes.

Lemma 4.1:

Assuming that the matrix KB is nonsingular and sliding mode exists on the intersections of the switching planes $s(x) = 0$ (4.22), then an equivalent control u_{eq} exists and is unique,

$$u_{eq} = (KB)^{-1}K[Ax + E_o f]. \tag{4.23}$$

Furthermore, when sliding mode occurs, the VSS (4.17), (4.18) is governed by

$$\dot{x} = (I - B(KB)^{-1}K)[Ax + E_o f] \tag{4.24}$$

$$Kx = 0. \tag{4.25}$$

Proof: Following the equivalent control method [38], (4.23) is the unique solution of u in the algebraic equation

$$\dot{s} = KAx + KBu + KE_o f = 0. \tag{4.26}$$

Substituting (4.23) into (4.17), (4.24) is obtained and the constraint (4.25) is the sliding mode condition $s(x) = 0$.

We call (4.24), (4.25), the equations of sliding mode. In the design of VSS with sliding mode, we consider the synthesis of the switching planes, that is, the choice of a matrix K such that the dynamical behavior of the VSS in sliding mode is desirable. Some synthesis procedures are developed in [15,17,37]. As it is illustrated in the example, the second design problem (the so-called reaching problem) in VSS is to guarantee that the trajectories move towards the intersections of the switching planes and that sliding mode exists. In [19,37], the reaching conditions and the existence conditions are derived from the conditions for the asymptotic stability in the large and in the small, respectively, of the subspace $s(x) = 0$ for the discontinuous feedback system (4.17), (418). To derive these conditions, it is necessary to construct Lyapunov functions for the discontinuous system

$$\dot{s} = KAx + KBu + KE_o f \tag{4.27}$$

Since there is no standard method to construct Lyapunov functions for arbitrary nonlinear systems, the existence and reaching conditions can be derived only when appropriate Lyapnov functions are found. Two cases when the Lyapunov functions exist are summarized in the next lemma [37].

<u>Lemma 4.2</u>:

If the matrices K and B are such that either

(i) there exist a positive definite symmetric matrix W and a matrix L which is diagonally dominant such that

$$L = - WKB \tag{4.28}$$

or

(ii) KB is symmetric,

then there exists Lyapunov functions which are quadratic forms of s.

Alternatively, the existence and reaching conditions can be found by utilizing the conditions for VSS with scalar control where there is only one switching plane $\sigma = 0$. If

$$\dot{\sigma}\sigma < 0, \tag{4.29}$$

that is, $\dot{\sigma}$ and σ have opposite signs, then the trajectories move toward $\sigma = 0$ and sliding mode exists. Two cases when it is possible to decouple the multivariable problem into m scalar problems is given in the next lemma [37].

<u>Lemma 4.3</u>:

If the matrices K and B are such that KB is diagonally dominant then the conditions on r_i of (4.21) such that the trajectories reach the switching plane $s_i = 0$ and sliding mode exists on $s_i = 0$ can be derived from

$$\dot{s}_i s_i < 0 \quad , \; i = 1,\ldots,m. \tag{4.30}$$

From the last two lemmas, we see that in order to derive the existence and reaching conditions, it is necessary that the matrix KB is of specific structures. The following lemma [37] enlarges the class of matrix KB.

Lemma 4.4:

The equations of sliding mode (4.24), (4.25) are invariant with respect to switching plane transformations

$$s^* = H_s s \tag{4.31}$$

and input transformations

$$u^* = H_u u \tag{4.32}$$

where H_s and H_u are m × m nonsingular matrices.

If the matrix B is known, then there exists matrix H_u (H_s) such that with respect to the new inputs (new switching planes), the new KB_o belongs to either cases in Lemma 4.2 and 4.3.

We note that, however, in the presence of parameter variations in B, there exists no transformation H_s or H_u that reduces to the cases of Lemma 4.2, 4.3. On the contrary, the hierarchy of controls method which is also given in [19,37], does not require KB to have any particular structure. Hence, this method is suitable for our purposes since all the VSS we considered later have parameter variations. To introduce this method, we use the following definitions.

Definition 4.1:

Let u_i be discontinuous on $s_i = 0$. If u_1 is designed so that sliding mode occurs on $s_1 = 0$ and assuming $s_1 = 0$, u_2 in designed so that sliding mode occurs on $s_2 = 0$, then the hierarchy of controls is $u_1 \rightarrow u_2$.

Definition 4.2:

If the hierarchy of controls is $u_1 \rightarrow u_2$, then the hierarchy of switching planes is $s_1 \rightarrow s_2$, that is, sliding mode first occurs on $s_1 = 0$ and then on the intersections of $s_1 = 0$ and $s_2 = 0$.

The hierarchy of controls method consists of the following steps.

Step 1. Suppose a hierarchy of controls is given, $u_1 \rightarrow u_2 \rightarrow \dots \rightarrow u_m$.

Step 2. We begin with the bottom switching plane $s_m = 0$ in the hierarchy by letting $i = m$.

Step 3. Suppose sliding mode occurs on the first $i-1$ switching planes, that is, $s_j = 0$, $j = 1,\dots,i-1$.† Solve for the equivalent control u_{eq}^{i-1} of the variable structure controls $u^{i-1} \equiv (u_1,\dots,u_{i-1})^T$ as a function of u_i and $u^{i+1} \equiv (u_{i+1},\dots,$ $(u_{i+1},\dots,u_m)^T$ from the algebraic equations $\dot{s}_j = 0$, $j = 1,\dots,i-1$. Since (4.17) is a linear system, u_{eq}^{i-1} is linear in x, u^{i+1}, f and u_i, that is,

$$u_{eq}^{i-1} = P_{i-1}x + Q_{i-1}u^{i+1} + T_{i-1}f + \alpha_i u_i \tag{4.33}$$

where matrices P_{i-1}, Q_{i-1} and T_{i-1} and scalar α_i depend on the first $i-1$ rows of matrices KA, KB and KE_o. We note that u^{i+1} is known since r_j, r_b^j, $j=i+1,\dots,m$ have been determined previously.

Step 4. For the i^{th} switching plane $s_i = 0$, solve for r_i such that

$$\dot{s}_i s_i < 0. \tag{4.34}$$

---

†For $i = 1$, we do not assume sliding mode occurs on any switching plane.

The resulting conditions on r_i are

$$\alpha_i^*[r_i^T|x|^\dagger + r_b^i] \le - \min_{u^{i+1},f} [p_i^T x + t_i^T f + q_i^T f + q_i^T u^{i+1}], \qquad \alpha_i^* \neq 0 \quad (4.35)$$

where r_b^i denotes the i^{th} component of r_b and the vectors p_i, t_i and q_i depend on the matrices P_{i-1}, Q_{i-1}, T_{i-1}, and scalar α_i, respectively and all of them depend on the i^{th} row of the matrices KA, KB and KE_o. In (4.35), we assume that the bounds on the components of the disturbance f are known. Since u^{i+1} is of the form (4.18), the minimizati n in (4.35) can be performed component wise, that is,

$$\alpha_i^* r_{ij} \le - \min_{r_{i+1,j},\ldots,r_{m,j}} [(p_{ij} + \sum_{k=i+1}^{m} q_{ik} r_{kj}) \operatorname{sgn} x_j], j = 1,\ldots,n \quad (4.36)$$

$$\alpha_i^* r_b^i \le - \min_{f, r_b^{i+1},\ldots,r_b^m} [t_i^T f + \sum_{k=i+1}^{m} q_{ik} r_b^k] \quad (4.37)$$

where p_{ij} and r_{ij} are the j^{th} component of p_i and r_i, respectively.

Step 5. Let $i = i-1$. If $i > 0$, go to Step 3, else stop.

Because of the inequality conditions (4.36), (4.37) in the above procedures, the hierarchy of controls method is suitable for systems with parameter variations. In this case, the maxima of the arguments on the right hand sides of (4.36), (4.37) with respect to the uncertainty parameters is minimized. It is sufficient to know only the bounds

---

†Let $x = (x_1,\ldots,x_n)^T$, then $|x| \equiv (|x_1|,\ldots,|x_n|)^T$.

of the variations in order to determine the switching gains r_i. For the hierarchy of controls specified in Step 1, the solution of r_i assume the worst case for r_j, r_b^j, $j = i+1,\ldots,m$ and in particular, r_1 assures sliding mode on $s_1 = 0$ assuming that sliding mode does not occur on $s_j = 0$, $j = 2,\ldots,m$. We will employ this method exclusively for the design of VSS with sliding mode in this study.

Thus far, we have considered the ideal behavior of VSS with sliding mode, that is, we assume that the switching of the variable structure control when the state x is on the switching planes is infinitely fast and consequently, the trajectories lie in the intersection of the switching planes when sliding mode occurs. In reality, due to switching delays, neglected time constants, hysteresis, etc., the control is switched at a finite frequency and the corresponding trajectories are in some vicinity of the switching planes. We refer to these trajectories as nonideal sliding mode. It is shown in [38] that nonideal sliding mode is "close" to the ideal sliding mode in the sense described in the next theorem [38].

<u>Theorem 4.1</u>:

Let the behavior of the VSS (4.17), (4.18) with nonideal switching devices be described by the system

$$\dot{x} = Ax + B\tilde{u} + E_o f \tag{4.38}$$

where the nonidealities are taken into account by the control $\tilde{u}$. Suppose that $\tilde{u}$ is such that $\tilde{x}(t)$, the motion of (4.38), exists and is in a Δ-neighborhood of $s(x) = 0$, that is,

$$|s(\tilde{x}(t))| \leq \Delta, \qquad t \in [t_1,\infty) \tag{4.39}$$

and Δ is a parameter depending on the type and size of the switching nonidealities. Then there exists a positive number γ such that

$$|x(t) - \tilde{x}(t)| \leq \gamma\Delta \tag{4.40}$$

for all $t \in [t_1,\infty)$ where $x(t)$ is the motion of the VSS (4.17), (4.18) in sliding mode characterized by the equations of sliding mode (4.24), (4.25).

If the trajectory moves to some Δ-neighborhood of $s(x) = 0$, then this theorem guarantees that it will remain in some vicinity of $s(x) = 0$. In other words, if the reaching conditions hold in the presence of switching nonidealities, then the motion is close to the ideal sliding mode. Furthermore, (4.40) implies that if ideal sliding mode $x(t)$ is asymptotically stable in the subspace $s(x) = 0$, then $\tilde{x}(t)$ is at least stable in the sense of Lyapunov.

In general, when the VSS (4.17), (4.18) is in sliding mode, its motion governed by the equations of sliding mode (4.24), (4.25) is influenced by the disturbance f. If there are uncertainties in the system papameters A, B, these parameter variations also affect the behavior of sliding mode. Let $A = A_o + \delta A$ and $B = B_o + \delta B$ when A_o, B_o denote the nominal values, then providing that $K(B_o + \delta B)$ is nonsingular, the equivalent control is

$$u_{eq} = [K(B_o + \delta B)]^{-1}K[(A_o + \delta A)x + E_o f] \tag{4.41}$$

and (4.24) becomes

$$\dot{x} = (I - (B_o + \delta B)[K(B_o + \delta B)]^{-1}K)[(A_o + \delta A)x + E_o f]. \tag{4.42}$$

In [39], the following conditions of invariancy of sliding mode with respect to plant parameter variations[†] and disturbances is given.

Lemma 4.5:

If

$$\mathrm{rank}[B_o + \delta B \,\vdots\, E_o] = \mathrm{rank}\,(B_o + \delta B) \tag{4.43}$$

and

$$\mathrm{rank}[B_o + \delta B \,\vdots\, \delta A] = \mathrm{rank}\,(B_o + \delta B) \tag{4.44}$$

then the equations of sliding mode (4.42), (4.25) do not depend on matrix δA and the disturbance f.

Conditions (4.43), (4.44) imply there exist matrices P and Q such that $\delta A = (B_o + \delta B)P$ and $E_o = (B_o + \delta B)R$, then (4.42) becomes

$$\dot{x} = (I - (B_o + \delta B)[K(B_o + \delta B)]^{-1}K)A_o x \tag{4.45}$$

which is clearly invariant with respect to plant parameter variations δA and disturbance f. We note that conditions (4.43), (4.44) resemble the conditions of arbitrary small sensitivities (3.17), (3.18) using high gain feedback. The relationships of VSS with sliding mode and high gain feedback systems are explored in details in the following section.

4.2. Sliding Mode and High Gain Feedback

A type of nonidealities in switching devices is the finite slope switching. In [40,41], it is demonstrated that an ideal switching device can be represented by an amplifier with an infinitely large gain and saturation. Thus, it is reasonable to conjecture that the behavior of VSS

[†]This means variations in matrix A only.

in the neighborhood of the switching planes is close to that of high gain feedback systems. This conjecture is verified in the following theorem.

Theorem 4.2:

In the VSS given by (4.17), (4.18), let $A = A_o$, $B = B_o$ and $f = 0$. If sliding mode occurs on $s = Kx = 0$ (4.22), then the motion of system (4.17), (4.18) is identical to the limiting motion of the slow subsystem (2.28) of the high gain feedback system (2.1), (2.7) as gain $g \to \infty$.

Proof: By applying the transformation M introduced in (2.9) in section 2.1, system (4.17) with $A = A_o$, $B = B_o$ and $f = 0$ becomes (2.51), (2.52). In the new coordinates, the equivalent control

$$u_{eq} = -(K_2B_2^o)^{-1}[(K_1A_{11}^o + K_2A_{21}^o)x_1' + (K_1A_{12}^o + K_2A_{22}^o)x_2'] \tag{4.46}$$

and the switching planes are

$$s = K_1x_1' + K_2x_2' = 0 \tag{4.47}$$

By substituting u_{eq} into (2.51), (2.52), the equations of sliding mode are given by

$$\dot{x}_1' = A_{11}^o x_1' + A_{12}^o x_2' \tag{4.48}$$

$$\dot{s} = 0 \tag{4.49}$$

and (4.47). Since K_2^{-1} exists, we can eliminate x_2' in (4.48) which gives

$$\dot{x}_1' = (A_{11}^o - A_{12}^o K_2^{-1} K_1)x_1'. \tag{4.50}$$

Hence, the equations of sliding mode can be represented by an $(n-m)^{th}$ order system (4.50). We call this the equivalent control system. If all the eigenvalues of KB_o have negative real parts, then (4.50) coincides with the slow subsystem (2.28) of the high gain feedback system (2.1) (2.7) as $g \to \infty$. For the case when KB_o has unstable eigenvalues, Lemma 4.4 shows that the equivalent control system is invariant with respect to switching plane transformation. Since KB_o is nonsingular, there exists a nonsingular matrix H_s in (4.31) such that $Re\lambda(H_s KB_o) < 0$. This completes the proof.

This theorem establishes the relationship between VSS and high gain feedback systems based upon the existence of sliding model on one hand and the asymptotic stability of the fast motions on the other hand. As gain $g \to \infty$, their motions in the null space of matrix K are identical whereas outside this null space, the motion of the high gain feedback system occurs infinitely fast in the range of B_o while the motion of VSS depend on the switching gains r_i in (4.21). The theorem also allows us to interpret the nonzero eigenvalues of the equation of sliding mode as the transmission zeros of the open loop system (4.17) with "output" s. Consequently, by application of the results of Section 2.3 and 2.4, the synthesis of switching planes becomes an n-m order pole placement problem or alternatively a quadratic minimization problems (problem "s").

In the presence of system parameter variations, we see from Section 3.1 that the stability of the fast motion is only robust in the sense of (3.20). From (4.41), the invertibility of the matrix $K\ (B_o + \delta B)$ is required for the equivalent control to be unique and consequently the equations of sliding mode to be unique. From Lemma 4.5 the sensitivities

AD-A057 462 ILLINOIS UNIV AT URBANA-CHAMPAIGN DECISION AND CONTROL LAB F/G 9/3
ANALYSIS AND SYNTHESIS OF HIGH GAIN AND VARIABLE STRUCTURE FEED--ETC(U)
NOV 77 K D YOUNG DAAB07-72-C-0259
UNCLASSIFIED DC-7 UILU-ENG-77-2247 NL

2 OF 3
AD
A057 462

of sliding mode with respect to parameter variations and disturbances depend on range conditions of δA, δB and E_o analogous to the conditions for high gain feedback systems given in Section 3.1. In view of the equivalence of sliding mode and the slow motion in high gain systems, we obtain the following new conditions of invariancy for sliding mode.

Theorem 4.3:

In the VSS given by (4.17), (4.18) let $A = A_o + \delta A$, $B = B_o + \delta B$ and suppose sliding mode exists on $s = Kx = 0$. Then if

$$\mathcal{R}(E_o) \subseteq \mathcal{R}(B_o) \tag{4.51}$$

$$\mathcal{R}(\delta A) \subseteq \mathcal{R}(B_o) \tag{4.52}$$

and

$$\mathcal{R}(\delta B) \subseteq \mathcal{R}(B_o) \tag{4.53}$$

then the equivalent control system is

$$\dot{x}_1' = (A_{11}^o - A_{12}^o K_2^{-1} K_1) x_1' \tag{4.54}$$

which is identical to (4.50).

Proof: If (4.51)-(4.53) holds, then using the same transformation M as in the proof of Theorem 4.2 (4.17) is transformed to

$$\dot{x}_1' = A_{11}^o x_1' + A_{12}^o x_2' \tag{4.55}$$

$$\dot{x}_2' = A_{21}^o x_1' + A_{22}^o x_2' + (B_2^o + \delta B_2)u + E_2^o f \tag{4.56}$$

where $ME_o = [0 \quad E_2^o]^T$. The equivalent control

$$u_{eq} = -(K_2(B_2^o + B_2))^{-1}[(K_1 A_{11}^o + K_2 A_{21}^o)x_1' + (K_1 A_{21}^o + K_2 A_{22}^o)x_2' + K_2 E_2^o f]. \tag{4.57}$$

By substituting u_{eq} into (4.56), we obtain $\dot{s} = 0$. Eliminating x_2' in (4.55) (4.54) is obtained.

Theorem 4.3 states the invariancy conditions in terms of the range of B_o while in Lemma 4.5, the condition on range of δB is not examined. We see that the range conditions (4.51)-(4.53) for the invariancy of sliding mode are identical to conditions (3.17)-(3.19) for high gain state feedback systems. Therefore, we conclude that VSS with sliding mode and high gain state feedback systems reject the same class of disturbances and are insensitive to the same class of parameter variations.

4.3. VSS with Luenburger Observer Feedback

In single input VSS in phase canonical form, the variable structure control is a function of the output and its derivative. There have been studies on the construction of estimators for variable structure feedback [42,43]. Also, attempts to design variable structure estimators have been made [44,45,46,47]. Unfortunately, extension of these results to multivariable VSS is not plausible yet we need them in the case when some of the states are inaccessible. Our approach is to construct the well known full order Luenburger observer and examine the behavior of the VSS with observer feedback.

The system considered is

$$\dot{x} = A_o x + B_o u \tag{4.58}$$

$$y = C_o x \tag{4.59}$$

where $x \in R^n$, $u \in R^m$ and the measured outputs $y \in R^r$ as in Chapter 3. We assume that A_o, B_o and C_o are known matrices. Then a full order Luenburger observer is of the form [48]

$$\dot{\hat{x}} = A_o\hat{x} + B_o u - H(y - \hat{y}) \tag{4.60}$$

$$\hat{y} = C_o\hat{x} \tag{4.61}$$

where $\hat{x} \in R^n$ is the observer states and $\hat{y} \in R^m$ is the observer outputs. Let matrix H be designed such that the error system

$$\dot{e}' = (A_o + HC_o)e' \tag{4.62}$$

with $$e' \equiv x - \hat{x} \tag{4.63}$$

be asymptotically stable and possesses desirable characteristics. Let the variable structure control be

$$u(x) = \hat{\Psi}^o\hat{x} + \Phi^o y \tag{4.64}$$

where

$$\hat{\Psi}^o = \hat{\Psi}\text{diag}(\text{sgn }\hat{x}_1,\ldots,\text{sgn }\hat{x}_n) \tag{4.65}$$

and $\hat{x}_i$ denotes the i^{th} component of $\hat{x}$,

$$\hat{\Psi}^T = (\hat{\Psi}_1,\ldots,\hat{\Psi}_m) \tag{4.66}$$

$$\hat{\Psi}_i = \begin{cases} \hat{r}_i & , \ \hat{s}_i(\hat{x}) > 0 \\ -\hat{r}_i & , \ \hat{s}_i(\hat{x}) < 0 \end{cases} \quad , \ i = 1,\ldots,m, \tag{4.67}$$

and $\hat{s}_i(\hat{x}) = 0$ are the switching planes. We define an m-vector $s(\hat{x})$ whose i^{th} component is denoted as $\hat{s}_i(\hat{x})$ by

$$\hat{s}(\hat{x}) = [\hat{s}_1(\hat{x}),\ldots,\hat{s}_m(\hat{x})]^T = K\hat{x} = 0 \tag{4.68}$$

Similarly,

$$\Phi^o = \Phi \operatorname{diag}(\operatorname{sgn} y_1,\ldots,\operatorname{sgn} y_r) \tag{4.69}$$

and y_i denotes the i^{th} component of y,

$$\Phi^T = (\phi_1,\ldots,\phi_m) \tag{4.70}$$

$$\phi_i = \begin{cases} \rho_i & , \hat{s}_i(\hat{x}) > 0 \\ & \qquad\qquad , \ i = 1,\ldots,m \\ -\rho_i & , \hat{s}_i(\hat{x}) < 0 \end{cases} \tag{4.71}$$

In (4.66), (4.67), $\hat{\psi}_i$ and $\hat{r}_i$ are n-vectors and in (4.70), (4.71), ϕ_i and ρ_i are r-vectors.

Lemma 4.6:

Let u be a variable structure control law given by (4.64), then there exists vectors $\hat{r}_i$ and p_i, $i = 1,\ldots,m$ such that the trajectories $\hat{x}(t)$ of the VSS (4.60), (4.64) move towards the intersection of the switching planes $\hat{s} = K\hat{x} = 0$ and sliding mode exists on $\hat{s} = 0$.

Proof: (4.60), (4.68) gives

$$\dot{\hat{s}} = K(A_o + HC_o)\hat{x} + KB_o u - KHy. \tag{4.72}$$

By considering y as measurable disturbances, we construct a variable structure control law of the form of (4.64) where y is "fed forward" with switching gains. For (4.64), the condition (4.35) of the hierarchy of controls method becomes

$$\alpha_i^*[\hat{r}_i^T|\hat{x}| + \rho_i^T|y|] \le - \min_{u^{i+1}}[p_i^T\hat{x} + t_i^T y + q_i^T u^{i+1}] \tag{4.73}$$

where the minimization in (4.73) can be performed component wise analogous to (4.36). This concludes the proof.

Since the switching planes (4.68) are defined in the observer state space $\hat{x}$, we have to examine the behavior of x(t) when sliding mode occurs on $s(\hat{x}) = 0$.

Lemma 4.7:

Let $t_r > 0$ be the time instant when sliding mode occurs on $\hat{s} = 0$ and let $t_e > 0$ be the time instant such that for some positive number Δ

$$|e'(t)| \le \frac{\Delta}{\|K\|} \tag{4.74}$$

for all $t \in [t_e, \infty)$, then the motion x(t) of (4.58) satisfies

$$|Kx(t)| \le \Delta \tag{4.75}$$

for all $t \in [\max(t_e,t_r), \infty)$.

Proof: Let e_s be defined as

$$e_s \equiv Ke'. \tag{4.76}$$

Assume that $t_e > t_r$. Then for $t \in [t_r,\infty)$, $Kx(t) = e_s(t)$. Since for all t, $|e_s(t)| \leq \|K\| |e(t)|$, (4.74) implies $|e_s(t)| \leq \Delta$ for all $t \in [t_e,\infty)$. Hence (4.75) is obtained. For the case when $t_r \geq t_e$, (4.76) implies that $Kx(t) = e_s(t) + K\hat{x}(t)$. By the triangular inequality and

$$|Kx(t)| \leq \Delta + |K\hat{x}(t)|, \qquad \text{for } t \in [t_e,\infty) \tag{4.77}$$

Since by definition $|K\hat{x}(t)| = 0$, $t \in [t_r,\infty)$, (4.77) results in (4.75). The Lemma is proved.

By defining $s = Kx$ as in (4.22), this lemma shows that $x(t)$ is confined to a Δ-neighborhood of $s = 0$ after sliding mode occurs on $\hat{s} = 0$. Thus, the behavior of VSS with observer feedback is analogous to that of VSS with switching nonidealities. The following theorem is a direct application of Theorem 4.1.

Theorem 4.4:

Let t_e, t_r and Δ be as defined in Lemma 4.7 and let $x(t)$ be the motion of the VSS (4.58), (4.64) and $\hat{x}(t)$ is the motion of the observer system (4.60),(4.61). Suppose that sliding mode exists on $\hat{s} = 0$, then there exists a positive number γ such that

$$|x(t) - \hat{x}(t)| \leq \gamma \Delta \tag{4.78}$$

for all $t \in [\max(t_e, t_r), \infty)$. Moreover,

$$\lim_{t\to\infty} |x(t) - \hat{x}(t)| = 0 \tag{4.79}$$

Proof: (4.78) is obtained by Theorem 4.1. Since $\lim_{t \to \infty} e(t) = 0$, hence $\Delta \to 0$ as $t \to \infty$ and this concludes the proof.

Clearly, if $\lim_{t \to \infty} \hat{x}(t) = 0$, then the VSS (4.58), (4.64) is asymptotically stable.

Lemma 4.8:

If sliding mode occurs on $\hat{s} = 0$, then the equations of sliding mode is given by

$$\dot{\hat{x}} = (I - B_o(KB_o)^{-1}K)[A_o\hat{x} - HC_oe'] \tag{4.80}$$

$$K\hat{x} = 0. \tag{4.81}$$

Proof: By setting the right hand side of (4.72) to zero and solving for u,

$$u_{eq} = -(KB_o)^{-1}[KA_o\hat{x} + KHC_oe'] \tag{4.82}$$

substituting (4.82) in (4.60), we have (4.80), (4.81).

To eliminate m variables in $K\hat{x} = 0$, we introduce the transformation M of (2.4) as in the proof of Theorem 4.2. Let

$$MH = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \tag{4.83}$$

where H_1 is an $(n-m) \times r$ matrix. The next theorem shows the "separation property" of VSS with observer feedback.

Theorem 4.5:

If sliding mode occurs on $\hat{s}(\hat{x}) = 0$ (4.68), then the equivalent control system is

$$\dot{\hat{x}}'_1 = (A^o_{11} - A^o_{12}K_2^{-1}K_1)\hat{x}'_1 + H_1C_oe' \tag{4.84}$$

$$\dot{e}' = (A_o + HC_o)e' \tag{4.85}$$

where $M\hat{x} = [\hat{x}_1, \hat{x}'_2]^T$ and $\hat{x}'_1$ is an (n-m) vector and M is as defined in (2.9).

Proof: Applying transformation M on (4.60) and following the proof of Theorem 4.2, we obtain (4.84).

From the block triangular structure of system (4.84), (4.85), we see that its eigenvalues are $\lambda(A^o_{11} - A^o_{12}K_2^{-1}K_1)$ and $\lambda(A_o + HC_o)$. Furthermore, the n-m eigenvalues $\lambda(A^o_{11} - A^o_{12}K_2^{-1}K_1)$ are identical to the eigenvalues of the equivalent control system (4.50) when the variable structure control feeds back all the states and switches on $s = Kx = 0$. Therefore, we can design matrix K as if all the states are accessible, separately from the design of matrix H. This property is analogous to the separation property of linear observer feedback systems [48]. We note that, in general, sliding mode does not occur on $s = Kx = 0$. Hence, the motion $x(t)$ is close to $\hat{x}(t)$ in the sense of (4.78),(4.79) when sliding mode occurs on $\hat{s} = 0$. But if $A^o_{11} - A^o_{12}K_2^{-1}K_1$ is a stable matrix, then the motion $x(t)$ approaches the subspace $s = Kx = 0$ and simultaneously moves toward the origin asymptotically as $t \to \infty$. What remains is to show the behavior of $x(t)$ before $\hat{x}(t)$ reaches $\hat{s} = 0$.

Lemma 4.9:

Let t_r be as defined in Lemma 4.7 and let $\alpha = \max \mathrm{Re}\lambda(A_o + HC_o)$ and $\beta = \max \mathrm{Re}\lambda(A^o_{11} - A^o_{12}K_2^{-1}K_1)$, then there exists positive numbers c_1, c_2 and c_3 which depend on initial conditions $\hat{x}(0)$ and $x(0)$ such that

$$|x(t)| \le c_1 + c_2e^{\alpha t} + c_3e^{\beta t} \tag{4.86}$$

for all $t \in [0, t_r)$.

Proof: Applying transformation M on (4.60) and if sliding mode does not occur on $\hat{s} = 0$,

$$\dot{\hat{x}}_1' = (A_{11}^o - A_{12}^oK_2^{-1}K_1)\hat{x}_1' + A_{12}^oK_2^{-1}\hat{s} + H_1C_oe'. \tag{4.87}$$

Since the variable structure control (4.64) is designed so that the motion $\hat{x}(t)$ moves towards $\hat{s}(\hat{x}) = K\hat{x} = 0$, there exists a positive number c_1 such that $|s(t)| \le c_1'$. Consequently, (4.87) implies that there exists positive numbers c_2', c_3' and c_4' such that

$$|\hat{x}(t)| \le c_2' + c_3'e^{\alpha t} + c_4'e^{\beta t}. \tag{4.88}$$

(4.85) implies that there exists a positive number c_5' such that

$$|e'(t)| \le c_5'e^{\alpha t} \tag{4.89}$$

Since $x \equiv \hat{x} + e'$ and $\hat{x} = P_1\hat{x}_1' + P_2\hat{s}$ for some matrices P_1 and P_2, by the triangular inequality and (4.88), (4.81), (4.86) is obtained

We see that since matrices $A_{11}^o - A_{12}^oK_2^{-1}K_1$ and $A_o + HC_o$ are stable, for bounded initial conditions, $\hat{x}(0)$ and $x(0)$, $|x(t)| - c_1$ is bounded by an exponentially decay time function.

4.4. Variable Structure Observer Design

The relationship of high gain feedback systems and VSS with sliding mode suggests the idea of an observer with a variable structure. Such an observer will possess similar insensitivity property of the

high gain observer when sliding mode occurs on appropriately chosen switching planes. The system we consider is the system (3.1), (3.2)

$$\dot{x} = (A_o + \delta A)x + (B_o + \delta B)u + E_o f \tag{4.90}$$

$$y = (C_o + \delta C)x. \tag{4.91}$$

Recall that A_o, B_o and C_o denote the nominal values and δA, δB, δC represent the parameter variations and f is the disturbance which is not measurable. Let a variable structure observer be of the form

$$\dot{\hat{x}} = A_o \hat{x} + B_o u - Hv \tag{4.92}$$

where v is a variable structure feedback discontinuous on the switching planes

$$s_e = C_o(x - \hat{x}) \equiv C_o e' = 0 \tag{4.93}$$

that is,

$$v_i(\hat{x},y) = \begin{cases} v_i^+(\hat{x},y) \quad , & s_e^i(e') > 0 \\ & \\ v_i^-(\hat{x},y) \quad , & s_e^i(e') < 0 \end{cases} \quad , \; i = 1,\ldots,r \tag{4.94}$$

where v_i and s_e^i are the i^{th} component of the r-vectors v and s_e, respectively. We note that unless $\delta C = 0$ in (4.91), we cannot determine the signs of $s_e^i(e')$ from the available measurements $y(t)$ and $\hat{x}(t)$

Although for a different reason, Theorem 3.3 shows that $\delta C = 0$ is also required in high gain observer design. Thus, for the design

variable structure observer, we make the assumption that the measurement matrix is known, $C = C_o$, that is,

$$\delta C = 0. \tag{4.95}$$

Moreover, we assume that the matrix C_oH is nonsingular. This assures the uniqueness of the equation of sliding mode. We now examine the behavior of system (4.92)-(4.94) when sliding mode occurs on $s_e = C_o e' = 0$.

Theorem 4.6:

Let $\hat{M}$ be the transformation defined in (3.38) and $e = [e_1 \;\; e_2]^T$ as in (3.37). Suppose sliding mode occurs on $s_e = C_o e' = 0$, then the equivalent control system is

$$\dot{e}_1 = (\hat{A}^o_{11} - \hat{A}^o_{12} C^{o^{-1}}_2 C^o_1) e_1 \tag{4.96}$$

if and only if

$$\mathcal{R}(\delta A) \subseteq \mathcal{R}(H) \tag{4.97}$$

$$\mathcal{R}(\delta B) \subseteq \mathcal{R}(H) \tag{4.98}$$

$$\mathcal{R}(E_o) \subseteq \mathcal{R}(H) \tag{4.99}$$

Proof: From (4.63), (4.90) and (4.92), we obtain

$$\dot{e}' = A_o e' + Hv + \delta Ax + \delta Bu + E_o f \tag{4.100}$$

Applying the transformation $\hat{M}$ to this equation, we obtain (3.39), (3.40). Solving for the equivalent control v_{eq} and substituting into (3.40), the

equivalent control system is

$$\dot{e}_1 = (\hat{A}^o_{11} - \hat{A}^o_{12}C^{o^{-1}}_2 C^o_1)e_1 + \delta\hat{A}_1 x + \delta\hat{B}_1 u + \hat{E}^o_1 f \qquad (4.101)$$

The conditions (4.97) - (4.99)are equivalent to $\delta\hat{A}_1 = 0$, $\delta\hat{B}_1 = 0$ and $\hat{E}^o_1 = 0$, respectively. This concludes the proof.

We note that (4.97)-(4.99) are the same range conditions (3.47)-(3.49) for insensitivity in high gain observer. Moreover, the eigenvalues of system (4.96) coincide with the eigenvalues of the slow subsystem of the high gain feedback system (3.36) as $\hat{g} \to \infty$. Theorem 3.2 gives the next lemma.

<u>Lemma 4.10</u>:

Let (A_o, C_o) be an observable pair, then there exists a matrix H such that $\lambda(\hat{A}^o_{11} - \hat{A}^o_{12}C^{o^{-1}}_2 C^o_1)$ are placed to q_j, $j = 1,\ldots,n-r$ where q_j are the prescribed locations of the eigenvalues of the equivalent control system of the VSS (4.92)-(4.94).

It remains to determine the feedback functions v^+_i, v^-_i in the variable structure feedback law (4.94). Using again the transformation $\hat{M}$,

$$\dot{s}_e = \hat{F}^o_1 e_1 + \hat{F}^o_2 s_e + C_o Hv + C_o \delta Ax + C_o \delta Bu + C_o E_o f \qquad (4.102)$$

where

$$\hat{F}^o_1 = C^o_1\hat{A}^o_{11} + C^o_2\hat{A}^o_{21} - \hat{F}^o_2 C^o_1 \qquad (4.103)$$

$$\hat{F}^o_2 = (C^o_1\hat{A}^o_{12} + C^o_2\hat{A}^o_{22})\, C^{o^{-1}}_2 \,. \qquad (4.104)$$

Since only s_e and u are accessible, we have to consider e_1, x and f as unmeasurable disturbances. Then a feasible variable structure control is

$$v(s_e,u) = \Psi_v^o s_e + \Pi_v u + \pi_b \tag{4.105}$$

where

$$\Psi_v^o = \Psi_v \operatorname{diag}(\operatorname{sgn} s_e^1,\ldots,\operatorname{sgn} s_e^r) \tag{4.106}$$

$$\Pi_v = \Pi \operatorname{diag}(\operatorname{sgn} u_1,\ldots,\operatorname{sgn} u_m) \tag{4.107}$$

$$\Pi^T = (\pi_1,\ldots,\pi_r) \tag{4.108}$$

and u_i, s_e^i denote the i^{th} component of m-vector u and r-vector s_e, respectively,

$$\pi_i = \begin{cases} \sigma_i & , s_e^i(e') > 0 \\ -\sigma_i & , s_e^i(e') < 0 \end{cases} \quad , \ i = 1,\ldots,r,b. \tag{4.109}$$

In (4.108), (4.109), π_i and σ_i, $i = 1,\ldots,r$ are m-vectors and π_b and σ_b are r-vectors.

Lemma 4.11:

Let v be a variable structure control given by (4.105), then there exists switching gain matrices Ψ_v and vectors σ_i, $i = 1,\ldots,r,b$ such that the motion $e'(t)$ moves towards the intersection of the switching planes $s_e = C_o e' = 0$ (4.93) and sliding mode exists on $s_e = 0$.

Proof: By considering u as measurable disturbances and e_1, x and f as unmeasurable disturbances and applying the hierarchy of controls method, inequalities similar to (4.35) and (4.73) are obtained for the variable structure control given by (4.105).

By this lemma, we have concluded the design of variable structure observer.

4.5. VSS With Variable Structure Observer Feedback

Thus far, we have shown that when all the states are accessible, VSS with sliding mode reject the same class of disturbances and parameter variations as in high gain state feedback systems. The same conclusion is made regarding variable structure observers and high gain observers. Therefore, it is expected that VSS with variable structure observer feedback, that is, a VSS with two variable structure loops, possesses the same insensitivity property as high gain system with high gain observer feedback studied in Section 3.4. In this class of variable sturcture feedback systems, u is a variable structure control for the system (4.90), (4.91) of the form,

$$u_i(\hat{x},y) = \begin{cases} u_i^+(\hat{x},y) & \text{if } \hat{s}_i(\hat{x}) > 0 \\ \\ u_i^-(\hat{x},y) & \text{if } \hat{s}_i(\hat{x}) < 0 \end{cases}, \quad i = 1,\dots,m \tag{4.110}$$

where the switching planes are as defined in (4.68),

$$\hat{s}(\hat{x}) = K\hat{x} = 0 \tag{4.111}$$

and $\hat{x}$ is the state of the variable structure observer (4.92)-(4.94). For

the same reason given in the last section, we assume $\delta C = 0$ in (4.91). We note that there are two sets of switching planes. The set $\hat{s}(\hat{x}) = 0$ is defined in the space of $\hat{x}$ and the set $s_e(e') = 0$ is defined in the space of e'. We now examine the behavior of the variable structure feedback system (4.90)-(4.94), (4.110), (4.111) when sliding mode occurs on $\hat{s} = 0$ and $s_e = 0$. We first determine the equivalent control.

Lemma 4.12:

If the $(m+r) \times (m+r)$ matrix

$$\Sigma \equiv \begin{bmatrix} KB_o & -KH \\ C_o\delta B & C_oH \end{bmatrix} \tag{4.112}$$

is nonsingular, then the equivalent control for sliding mode occurs on $\hat{s} = 0$ and $s_e = 0$ is unique and is given by

$$\begin{bmatrix} u \\ v \end{bmatrix}_{eq} = - \begin{bmatrix} KB_o & -KH \\ C_o\delta B & C_oH \end{bmatrix}^{-1} \begin{bmatrix} KA_o & 0 \\ C_o\delta A & C_o(A_o + \delta A) \end{bmatrix} \begin{bmatrix} \hat{x} \\ e' \end{bmatrix} + \begin{bmatrix} 0 \\ C_oE_o \end{bmatrix} f \tag{4.113}$$

Proof: If Σ is nonsingular, then the equivalent (4.113) can be solved uniquely in the simultaneous algebraic equations $\dot{\hat{s}} = 0$ and $\dot{s}_e = 0$.

The corresponding conditions for (4.112) in high gain observer feedback systems is (3.82).

Theorem 4.7:

Let M and $\hat{M}$ be the transformations defined in (2.9) and (3.38) respectively and $e = [e_1 \;\; e_2]^T$ as in (3.37). Denote $[\hat{x}_1' \;\; \hat{x}_2']^T = M\hat{x}$. Let

H_1 be as defined in (4.83) and N as in (2.19). Suppose that $\sum$ in (4.112) is invertible and sliding mode occurs on $\hat{s} = 0$ and $s_e = 0$, then the equivalent control system is represented by

$$\dot{e}_1 = (\hat{A}^o_{11} - \hat{A}^o_{12}C^{o^{-1}}_2 C^o_1)e_1 \tag{4.114}$$

$$\dot{\hat{x}}_1 = (A^o_{11} - A^o_{12}K_2^{-1}K_1)\hat{x}_1 + H_1(C_oH)^{-1}C_oA_oNe_1 \tag{4.115}$$

if and only if

$$R(E_o) \subseteq R(B_o),\ R(E_o) \subseteq R(H) \tag{4.116}$$

$$R(\delta A) \subseteq R(B_o),\ R(\delta A) \subseteq R(H) \tag{4.117}$$

and

$$R(\delta B) \subseteq R(B_o),\ R(\delta B) \subseteq R(H) \tag{4.118}$$

<u>Proof</u>: Let

$$\Omega \equiv (KB_o + KH(C_oH)^{-1}C_o\delta B), \tag{4.119}$$

then (4.113) can be written as

$$u_{eq} = \Omega^{-1}[(KH(C_oH)^{-1}C_o\delta A - KA_o)\hat{x}$$

$$- KH(C_oH)^{-1}(C_o(A_o + \delta A)e' + C_oE_of)] \equiv U_1\hat{x} + U_2e' + U_3f \tag{4.120}$$

$$v_{eq} = -(C_oH)^{-1}\{[C_o\delta B\Omega^{-1}(KH(C_oH)^{-1}C_o\delta A - KA_o) + C_o\delta A]\hat{x}$$

$$- [C_o\delta B\Omega^{-1}(I - KH(C_oH)^{-1}C_o][(A_o + \delta A)e' + E_of] \equiv V_1\hat{x} + V_2e' + V_3f \tag{4.121}$$

Ω is nonsingular by the assumption that $\sum$ is nonsingular. Substituting u_{eq} and v_{eq} for u and v respectively in (4.92) and (4.100), the equation of sliding mode is obtained,

$$\dot{\hat{x}} = (A_o + B_oU_1 - HV_1)\hat{x} + (B_oU_2 - HV_2)e' + (B_oU_3 - HV_3)f \tag{4.122}$$

$$K\hat{x} = 0 \tag{4.123}$$

$$\dot{e}' = (\delta BU_1 + HV_1)\hat{x} + (A_o + \delta BU_2 + HV_2)e' + (E_o + \delta BU_3 + HV_3)f \tag{4.124}$$

$$C_oe' = 0. \tag{4.125}$$

The conditions $\mathcal{R}(\delta B) \subseteq \mathcal{R}(H)$, $\mathcal{R}(\delta A) \subseteq \mathcal{R}(H)$ and $\mathcal{R}(E_o) \subseteq \mathcal{R}(H)$ are equivalent to the identities $H(C_oH)^{-1}C_o\delta B = \delta B$, $H(C_oH)^{-1}C_o\delta A = \delta A$ and $H(C_oH)^{-1}C_oE_o = E_o$, respectively which simplifies (4.122), (4.124) to

$$\dot{\hat{x}} = [A_o + (B_o + \delta B)U_1' + \delta A]\hat{x} + [(B_o + \delta B)U_2' + H(C_oH)^{-1}C_oA_o + \delta A]e'$$
$$+ [(B_o + \delta B)U_3' + E_o]f \tag{4.126}$$

$$\dot{e}' = [\delta BU_1']\hat{x} + [A_o + \delta BU_2' - H(C_oH)^{-1}C_oA_o - \delta A]e'$$
$$+ [2E_o + \delta BU_3']f \tag{4.127}$$

where

$$U_1' = [K(B_o + \delta B)]^{-1}(K\delta A - KA_o) \tag{4.128}$$

$$U_2' = - [K(B_o + \delta B)]^{-1}(KH(C_oH)^{-1}C_oA_o + K\delta A) \tag{4.129}$$

$$U_3' = - [K(B_o + \delta B)]^{-1}KE_o. \tag{4.130}$$

Applying the transformation M on (4.126) and eliminating $\hat{s}$, we obtain

$$\dot{\hat{x}}_1' = (A_{11}^o - A_{12}^o K_2^{-1} K_1)\hat{x}_1' + M_1(\delta BU_1' + \delta A)Q\hat{x}_1'$$

$$+ M_1 H(C_o H)^{-1} C_o A_o e' + M_1(\delta BU_2' + \delta A)e' \qquad (4.131)$$

$$+ M_1(\delta BU_3' + E_o)f$$

where $M_1 B_o = 0$ and $KQ = 0$ as defined in (2.21) and (3.53), respectively. Applying the transformation $\hat{M}$ on (4.127) and eliminating s_e, we obtain

$$\dot{e}_1 = (A_{11}^o - A_{12}^o C_2^{o^{-1}} C_1^o)e_1 + \hat{M}_1(\delta BU_2' - \delta A)Ne_1 + \hat{M}_1 \delta BU_1' x + \hat{M}_1(2E_o + \delta BU_3')f \qquad (4.132)$$

whre $\hat{M}_1 H = 0$ and $C_o N = 0$ as in (3.94) and (2.19), respectively. The conditions (4.116)-(4.118) are equivalent to $\hat{M}_1 E_o = 0$, $M_1 E_o = 0$; $\hat{M}_1 \delta A = 0$, $M_1 \delta A = 0$; and $\hat{M}_1 \delta B = 0$, $M_1 \delta B = 0$ respectively. By eliminating s_e from e' in (4.131), the equivalent control system (4.114), (4.115) is obtained. This concludes the proof.

We note that conditions (4.116)-(4.118) coincide with the range conditions (3.78)-(3.80) for insensitivity in high gain observer feedback systems. Since system (4.114),(4.115) is in block triangular form, its eigenvalues are $\lambda(\hat{A}_{11}^o - \hat{A}_{12}^o C_2^{o^{-1}} C_1^o)$ and $\lambda(A_{11}^o - A_{12}^o K_2^{-1} K_1)$. From (4.54) we see that $\lambda(A_{11}^o - A_{12}^o K_2^{-1} K_1)$ are the eigenvalues of the equivalent control system when the switching planes are $s = Kx = 0$ whereas from (4.96), $\lambda(\hat{A}_{11}^o - \hat{A}_{12}^o C_2^{o^{-1}} C_1^o)$ are the eigenvalues of the variable structure observer in sliding mode. This property is analogous to the separation property of VSS with Luenburger observer feedback.

In general, sliding mode does not exist on $s(x) = Kx = 0$. However, it can be shown that the motion $x(t)$ of (4.90) is close to $\hat{x}(t)$ when sliding mode occurs on $\hat{s} = 0$ and $s_e = 0$.

Lemma 4.13:

Let $t_r > 0$ be the time instant when sliding mode occurs on $\hat{s} = 0$ and $s_e = 0$ and let $t_e > 0$ be the time instant such that given a positive number Δ

$$|e_1(t)| \leq \| KN \|^{-1} \Delta \tag{4.133}$$

for all $t \in [t_e, \infty)$ where e_1 in the vector defined in Theorem 4.7 and matrix N satisfies $C_o N = 0$ as in (3.94). Then the motion $x(t)$ of the VSS with variable structure feedback satisfies

$$|Kx(t)| \leq \Delta \tag{4.134}$$

for all $t \in [\max(t_e, t_r), \infty)$.

Proof: Analogous to the decomposition in (2.18), e' can be written as

$$e' = Ne_1 + H(C_o H)^{-1} s_e \, . \tag{4.135}$$

Assume that $t_e > t_r$, then for $t \in [t_r, \infty)$ $Kx(t) = KNe_1(t)$. Since for all t, $|Kx(t)| \leq \| KN \| \; |e_1(t)|$, (4.133) implies (4.134). If

$t_r > t_e$, then for $t \in [t_e, \infty)$, $Kx(t) = KNe_1(t) + KH(C_oH)^{-1}s_e + K\hat{x}(t)$. By the triangular inequality and (4,133)

$$|Kx(t)| \le \Delta + \|KH(C_oH)^{-1}\| \, |s_e(t)| + |\hat{s}(t)| \tag{4.136}$$

for $t \in [t_e, \infty)$. Since $|s_e(t)| = 0$ and $|\hat{s}(t)| = 0$ for $t \in [t_r, \infty)$, by definition, (4.136) results in (4.134). The lemma is proved.

Be defining $s(x) = Kx$ as in (4.22), we see that $x(t)$ is in a Δ-neighborhood of $s(x) = 0$ after sliding mode occurs on $\hat{s}(\hat{x}) = 0$ and $s_e(e') = 0$. This phenomenon is analogous to that in VSS with Luenburger observer feedback and can be considered to be caused by switching non-idealities. Applying Theorem 4.1, we express $x(t)$ as an approximation to $\hat{x}(t)$ in the following Theorem.

Theorem 4.8:

Let t_e, t_r and Δ be as defined in Lemma 4.13 and let $x(t)$ be the motion of the VSS with variable structure feedback given by (4.90)-(4.94), (4.110)-(4.111) and let $\hat{x}(t)$ be the motion of the variable structure observer system (4.90)-(4.94). Assume that $\mathrm{Re}\lambda(\hat{A}^o_{11} - \hat{A}^o_{12}C^{o^{-1}}_2 C^o_1) < 0$. Suppose that sliding mode exists on $\hat{s} = K\hat{x} = 0$ and $s_e = C_oe' = 0$, then there exists a positive number γ such that

$$|x(t) - \hat{x}(t)| \le \gamma\,\Delta \tag{4.137}$$

for all $t \in [\max(t_e, t_r), \infty)$. Moreover,

$$\lim_{t \to \infty} |x(t) - \hat{x}(t)| = 0 \tag{4.138}$$

Proof: (4.137) is a direct consequence of (4.134) and Theorem 4.1. Since the matrix $\hat{A}^o_{11} - \hat{A}^o_{12}C^{o^{-1}}_2 C^o_1$ is stable, $\lim_{t \to \infty} e_1(t) = 0$. Hence $\Delta \to 0$ as $t \to \infty$ and (4.137) becomes (4.138).

From the equivalent control system (4.114), (4.115) we conclude that of $A^o_{11} - A^o_{12}K_2^{-1}K_1$ is a stable matrix. Thus $\lim_{t \to \infty} \hat{x}(t) = 0$ and Theorem 4.8 implies that the motion x(t) goes to zero asymptotically as $t \to \infty$. If Theorem 4.7 holds, then $\hat{x}_1(t)$, the motion in the subspace $K\hat{x} = 0$ is invariant with respect to parameter variations and disturbances. Since x(t) is $O(\Delta)$ close to $\hat{x}(t)$ (in the sense of (4.137)) and is in the $O(\Delta)$ neighborhood of $s(x) = 0$ (in the sense of (4.134)), we conclude that for $t \in [\max(t_e, t_r), \infty)$, the sensitivities of x(t) with respect to parameter variations and disturbances are $O(\Delta)$.

Analogous to Lemma 4.9, we now show the boundedness of the motion x(t) and e'(t) before sliding mode occurs on $\hat{s} = 0$ and $s_e = 0$.

Lemma 4.14:

Let t_r be as defined as Lemma 4.13 and let $\alpha = \max \mathrm{Re}\lambda(\hat{A}^o_{11} - \hat{A}^o_{12}C^{o^{-1}}_2 C^o_1)$ and $\beta = \max \mathrm{Re}\lambda(A^o_{11} - A^o_{12}K_2^{-1}K_1)$. Suppose that conditions (4.116)-(4.118) hold then there exist positive numbers c_1, c_2 and c_3 which depend on initial conditions x(0) and $\hat{x}(0)$ such that

$$|x(t)| \leq c_1 + c_2e^{\alpha t} + c_3e^{\beta t} \tag{4.139}$$

for all $t \in [0, t_r)$.

Proof: Applying transformation M and $\hat{M}$ as defined in Theorem 4.7 to (4.90), (4.100). Under the conditions of this Lemma, if sliding mode

does not occur on $\hat{s} = 0$ and $s_e = 0$,

$$\dot{\hat{x}}_1' = (A_{11}^o - A_{12}^o K_2^{-1} K_1)\hat{x}_1' + A_{12}^o K_2^{-1} \hat{s} - H_1 v \tag{4.140}$$

$$\dot{e}_1 = (\hat{A}_{11}^o - \hat{A}_{12}^o C_2^{o^{-1}} C_1^o) e_1 + \hat{A}_{12}^o C_2^{o^{-1}} s_e \tag{4.141}$$

Since the variable structure control u and v are designed such that the motion $\hat{x}(t)$ moves towards $\hat{s}(\hat{x}) = K\hat{x} = 0$ and the motion $e'(t)$ moves toward $s_e(e') = C_o e' = 0$, there exist positive numbers c_1' and c_2' such that $|\hat{s}(t)| \le c_1'$ and $|s_e(t)| \le c_2'$. Moreover, since $v(t)$ is bounded, consequently, from (4.140)

$$|\hat{x}_1'(t)| \le c_3' + c_4' e^{\beta t} \tag{4.142}$$

and from (4.141)

$$|e(t)| \le c_5' + c_6' e^{\alpha t} \tag{4.143}$$

Since $x = \hat{x} + e'$ and $\hat{x} = P_1 \hat{x}_1' + P_2 \hat{s}$, $e' = P_3 e_1 + P_4 s_e$ for some matrices P_1, P_2, P_3 and P_4, by the triangular inequality and (4.142), (4.143), we obtain (4.139).

We note that since the matrices $\hat{A}_{11}^o - \hat{A}_{12}^o C_2^{o^{-1}} C_1^o$ and $A_{11}^o - A_{12}^o K_2^{-1} K_1$ are stable, $|x(t)| - c_1$ is bounded by an exponentially decaying time function for bounded initial conditions $x(0)$ and $\hat{x}(0)$.

In the last section, the variable structure control v of (4.105) in the variable structure observer includes feedforward terms of u. If u is also a variable structure control, these terms are not necessary. Since the accessible variables include $\hat{x}$, s_e, feasible variable structure control schemes for u and v are

$$v(\hat{x}, s_e) = \tilde{\Psi}_v \hat{x} + \Psi_v^o s_e + \pi_b \tag{4.144}$$

and

$$u(\hat{x}, s_e) = \hat{\Psi}^o \hat{x} + \Psi_u^o s_e + \hat{\psi}_b \tag{4.145}$$

where Ψ_v^o, $\hat{\Psi}^o$ and π_b are as in (4.106), (4.65) and (4.105), respectively and $\tilde{\Psi}_v$, Ψ_u^o and $\hat{\psi}_b$ are defined as follows.

$$\tilde{\Psi}_v = \tilde{\Psi}\, \text{diag}(\text{sgn}\ \hat{x}_1, \ldots, \text{sgn}\ \hat{x}_n) \tag{4.146}$$

$$\tilde{\Psi}^T = (\tilde{\psi}_1, \ldots, \tilde{\psi}_r) \tag{4.147}$$

$$\tilde{\phi}_i = \begin{cases} \delta_i & , s_e^i(e') > 0 \\ -\delta_i & , s_e^i(e') < 0 \end{cases}, \quad i = 1, \ldots, r \tag{4.148}$$

$$\Psi_u^o = \Psi_u\, \text{diag}(\text{sgn}\ s_e^1, \ldots, \text{sgn}\ s_e^r) \tag{4.149}$$

and

$$\hat{\psi}_b = \begin{cases} \delta_b & , \quad \hat{s}_i(\hat{x}) > 0 \\ -\delta_b & , \quad \hat{s}_i(\hat{x}) < 0 \end{cases}, \quad i = 1, \ldots, m. \tag{4.150}$$

In the above equations, $\tilde{\psi}_i$ and δ_i, $i = 1, \ldots, r$ are n-vectors, $\hat{\psi}_b$ and δ_b are m-vectors.

<u>Lemma 4.15</u>:

Let v and u be variable structure controls given by (4.144) (4.145), then there exists switching gain matrices Ψ_u, Ψ_v and vectors $\hat{r}_i$, σ_j, δ_j, σ_b and δ_b, $i = 1, \ldots, m$, $j = 1, \ldots, r$, such that $e'(t)$ and $\hat{x}(t)$

move towards the intersections of the switching planes $s_e = C_o e' = 0$ and $\hat{s} = K\hat{x} = 0$, respectively. Moreover, sliding mode exists on $s_e = 0$ and $\hat{s} = 0$.

Proof: By considering e_1, x and f as unmeasurable disturbances and applying the hierarchy of controls method, inequalities similar to (4.35) and (4.73) are obtained for the variable structure controls given by (4.144) and (4.145).

4.6. Regulation by VSS

We conclude our study on VSS with sliding mode by considering the problem of regulation of some specified variables $z = D_o x$ in the presence of system parameter variations and disturbances using variable structure controls. We show that the conditions for the regulation of z in VSS are the same as when high gain feedback control is employed. The system we consider is system (4.90), (4.91), the same one in (3.3). Let $z = D_o x$ be the variables to be regulated. We first assume that all the states are accessible and let u be the variable structure control given by (4.18) such that x(t) moves toward $s(x) = Kx = 0$ (4.22) and sliding mode exists on $s = 0$. Then the condition for regulation of z(t) is given in the following theorem.

Theorem 4.9:

Let $t_r > 0$ be the time instant when sliding mode occurs on $s(t) = Kx = 0$ and let $K = [K_f K_s \;\; K_f]$ then $z(t) = 0$ for $t \geq t_r$ if and only if

$$\mathrm{Re}\lambda(Z_1^o + \delta Z_1 + \delta B_1 L) < 0 \qquad (4.151)$$

and

$$<Z_1^o + \delta Z_1 + \delta B_1 L | \mathcal{R}(M_1[I - \delta B(K(B_o + \delta B))^{-1}K]E_o)> \subseteq \mathcal{N}(D_o Q) \tag{4.152}$$

where the notations used are the same as in Theorem 3.5.

Proof: Applying the transformation M of (2.9), system (4.90) is of the form of (3.9), (3.10). The resulting equivalent control system for sliding mode on $s = 0$ is

$$\dot{x}_1' = (Z_1^o + \delta Z_1 + \delta B_1 L)x_1' + (M_1[I - \delta B(K(B_o + \delta B))^{-1}K]E_o)f \tag{4.153}$$

Since by definition $s = \omega$, from Theorem 3.5,

$$z = D_o Q x_1' + D_o B_o (KB_o)^{-1} s \tag{4.154}$$

Hence $z(t) = D_o Q x_1'(t)$, $t \geq t_r$. Using Wonham's result [33], the condition (4.152) is obtained. Condition (4.151) assures the VSS in sliding mode is asymptotically stable. This proves the Theorem.

We note that conditions (4.151) and (4.152) are the same conditions (3.33), (3.55) for the regulation of $z(t)$ using high gain state feedback. For the same reasoning given in Section 3.5, (3.152) is replaced by

$$\text{rank } K^T = \text{rank}[D_o^T \quad K^T] \tag{4.155}$$

or equivalently there exists matrix P such that

$$D_o = PK \tag{4.156}$$

An interpretation of (4.155) is that regulation of $z(t)$ is achieved once

slide mode occurs on $s = 0$ which can be seen by multiplying on both sides of (4.156) with $x(t)$.

In the case when only $y(t)$ of (4.91) is accessible, we assume that $\delta C = 0$ and the matrices δA, δB and E_o satisfy (4.97) - (4.99). Let u and v be variable structure controls given by (4.144), (4.145) such that $\hat{x}(t)$ moves toward $\hat{s} = 0$ (4.68) and $e'(t)$ moves toward $s_e = 0$ (4.93) and sliding mode exists on $\hat{s} = 0$ and $s_e = 0$. Theorem 4.6 shows that under the assumptions we have made, the motion $e'(t)$ is invariant with respect to parameter variations and disturbances in the subspace $s_e = 0$. We now assume that this sliding mode in $s_e = 0$ is asymptotically stable.

Theorem 4.10:

Let $t_r > 0$ be the time instant when sliding mode occurs on $\hat{s} = K\hat{x} = 0$ and $s_e = C_o e = 0$ and let α be as defined in Lemma 4.14,

$$Z \equiv A^o_{11} - A^o_{12}K_2^{-1}K_1 + M_1(\delta BU_1' + \delta A)Q \tag{4.157}$$

and $\beta = \max \lambda(Z)$, then there exist positive numbers c_1 and c_2 such that

$$|z(t)| \leq c_1 e^{\alpha t} + c_2 e^{\beta t} \tag{4.158}$$

for all $t \in [t_r, \infty)$ and $\lim_{t \to \infty} z(t) = 0$ if and only if

$$\mathrm{Re}\lambda(A^o_{11} - A^o_{12}K_2^{-1}K_1 + M_1(\delta BU_1' + \delta A)Q) < 0 \tag{4.159}$$

and

$$\langle (A^o_{11} - A^o_{12}K_2^{-1}K_1 + M_1(\delta BU_1' + \delta A)Q) | \mathcal{R}[M_1(I-\delta B(K(B_o + \delta B))^{-1}K)E_o] \rangle$$
$$\subseteq \mathcal{N}(D_oQ) \tag{4.160}$$

where the notations used are the same as in (4.128)-(4.131).

Proof: System (4.131), (4.132) represents the equivalent control system of the VSS with switching planes $\hat{s} = 0$ and $s_e = 0$. The conditions (4.97)-(4.99) are equivalent to $\hat{M}_1 \delta B = 0$, $\hat{M}_1 \delta A = 0$ and $\hat{M}_1 E_o = 0$. Hence (4.132) is reduced to (4.96) and there exists positive c_1' such that $|e_1(t)| \le c_1' e^{\alpha t}$, $t \in [t_r, \infty)$. Since $\hat{x}$ can be decomposed analogously as in (2.18) into

$$\hat{x} = Q\hat{x}_1' + B_o(KB_o)^{-1}\hat{s} \tag{4.161}$$

and e' can be expressed as in (4.135), hence

$$z(t) = D_o N e_1(t) + D_o Q \hat{x}_1'(t) \tag{4.162}$$

From (4.131), the solution $\hat{x}_1'(t)$ is given by

$$\hat{x}_1'(t) = \hat{x}_{1f}'(t) + \hat{x}_{1h}(t) \tag{4.163}$$

where $\hat{x}_{1f}'(t)$ is the component due to f(t). Wonham's result [33] shows that $\hat{x}_{1f}'(t) = 0$ if and only if (4.160) holds. The component $x_{1h}(t)$ is due to $\hat{e}_1'(t)$ and "initial condition" $\hat{x}_1'(t_r)$ and there exist positive c_2' and and c_3' such that

$$|\hat{x}_{1h}(t)| \le c_2' e^{\alpha t} + c_3' e^{\beta t}$$

Applying the triangular inequality to (4.162), (4.158) is obtained. By the stability assumption on the observer and (4.159), $\alpha < 0$ and $\beta < 0$, (4.158) implies $\lim_{t\to\infty} z(t) = 0$. The theorem is proved.

We note that conditions (4.159), (4.160) are equivalent to the conditions (3.101), (3.102) in Theorem 3.11 for the regulation of $z(t)$ using high gain observer feedback. For the same reasoning given in Section 3.5, (4.160) is replaced by (4.155). If D_o satisfies (4.155) or equivalently (4.156), then from (4.134),

$$|z(t)| \leq \| P \| \Delta \tag{4.164}$$

for $t > \max(t_e, t_r)$ and in view of (4.138), $\lim_{t \to \infty} z(t) = 0$. By comparing Theorem 4.9 and 4.10, we see that if $y(t)$ is the only information available, $z(t)$ is regulated asymptotically whereas it is zeroed as sliding mode occurs on $s(x) = Kx = 0$ in the case when all the states are accessible.

Finally, we observe that the synthesis problems K and H discussed in Section 3.6 are encountered again in VSS with sliding mode. They become the synthesis of the switching planes $s = Kx = 0$ and $\bar{s}_e = H^T\bar{e} = 0$. The switching planes $\bar{s}_e = 0$ are constructed for the dual of the VSS (4.100) with $\delta A = 0$, $\delta B = 0$ and $f = 0$, that is,

$$\dot{\bar{e}} = A_o^T\bar{e} + C_o^T\bar{v} \tag{4.165}$$

where the components $\bar{v}_i$ of the variable structure control $\bar{v}$ is discontinuous on the components $\bar{s}_e^i(\bar{e}) = 0$ of $\bar{s}_e(\bar{e}) = 0$. We note that for VSS, it is not required that the matrices KB_o and C_oH be stable as in the high gain feedback case. Instead, they should be invertible. This change in specification, however, does not change the solvability of the synthesis problems K and H of Section 3.6.

In this chapter, we have established the relationship between VSS and high gain feedback systems. It is shown that the insensitivity property of sliding mode in VSS and the slow motion in high gain systems depends on the same range conditions. In other words, these two types of feedback systems reject the same class of disturbances and parameter variations. Furthermore, sliding mode coincides with the limiting motion of the slow subsystem as the loop gains tend to infinity. We emphasize that the behaviors of VSS and high gain system are similar only in the subspace defined by the intersections of the switching planes which coincides with the null space of the high gain feedback matrix. In high gain system, the motion outside this subspace is extremely fast due to the large gain factor and its behavior depends on the constant feedback matrix. On the other hand, the motion before the occurrence of sliding mode is not necessarily fast and its behavior is determined by the variable structure control. In the case when reduced order models are used in the design of high gain systems and VSS, the capability to reach this subspace is susceptible to the neglected small time constants if the reaching motion is forced to be extremely fast by the use of large feedback gains. However, since high gain is not necessary in the design of variable structure control in VSS, reaching can be guaranteed. These aspects regarding robustness property of high gain systems and VSS with respect to actuator and sensor dynamics are examined in the next chapter.

CHAPTER 5

APPLICATIONS

5.1 Systems with Actuator and Sensor Dynamics

An important consideration in the realization of high gain feedback systems is the effects of actuator and sensor dynamics. In many control applications, these dynamics are neglected in order to reduce the complexity of the system model. Since the insensitivity property of high gain feedback system depends solely on the system structure, it is necessary to examine the effects on the actual system if a reduced order model is employed in the high gain feedback design. In this section, we shall examine only the effects of actuator and sensor dynamics. We assume all the states are measurable and there are no system parameter variations and disturbances. The system

$$\dot{x} = A_o x + B_o u \tag{5.1}$$

is considered where $x \in R^n$ and $u \in R^m$. This is the system model we have been using in the previous chapters and the high gain feedback is written as

$$u = gKx. \tag{5.2}$$

In the presence of actuator and sensor dynamics, (5.2) becomes

$$\upsilon = gK\chi \tag{5.3}$$

where $\chi \in R^n$ is the state vector of the sensor system

$$\varepsilon_1 \dot{\chi} = R_o \chi + S_o x \tag{5.4}$$

and the control variables are the states of the actuator system,

$$\varepsilon_2 \dot{u} = P_o u + Q_o \upsilon \tag{5.5}$$

We call χ, the sensor variables and υ, the actuator variables. In (5.4), (5.5), ε_1 and ε_2 are small positive scalars and matrices R_o and P_o are stable matrices. Without loss of generality, we assume that $P_o^{-1}Q_o$ and $R_o^{-1}S_o$ are invertible. If the feedback loop (5.3) is opened, then for an arbitrary continuous function $\upsilon(t)$, (5.5) gives

$$u(t) = -P_o^{-1} Q_o \upsilon(t) + 0(\varepsilon_2) \tag{5.6}$$

for $t \geq t_2$ where $t_2 = |\max\lambda(P_o)|^{-1}\varepsilon_2 \ln\varepsilon_2$ and

$$\chi(t) = -R_o^{-1} S_o x(t) + 0(\varepsilon_1) \tag{5.7}$$

for $t \geq t_1$ where $t_1 = |\max\lambda(R_o)|^{-1}\varepsilon_1 \ln\varepsilon_1$. Thus the parameters ε_1 and ε_2 determine the transient periods of the open loop behaviors of the sensor and actuator. In particular, if $-P_o^{-1}Q_o = I_m$ and $-R_o^{-1}S_o = I_n$, then $u(t)$ and $\chi(t)$ are $0(\varepsilon_2)$ and $0(\varepsilon_1)$ "close" to $\upsilon(t)$ and $x(t)$, respectively.

We consider now the behavior of the high gain feedback system with actuator and sensor dynamics, (5.1), (5.3)-(5.5). First, we assume that $\varepsilon_1^{-1} \gg g$ and $\varepsilon_2^{-1} \gg g$, the results are stated in the following theorem.

<u>Theorem 5.1</u>:

Suppose ε_1, ε_2 and g are such that, for some positive number α

$$\lim_{\substack{\varepsilon_1 \to 0 \\ g \to \infty}} \frac{g}{\varepsilon_1} = 0, \quad \lim_{\substack{\varepsilon_2 \to 0 \\ g \to \infty}} \frac{g}{\varepsilon_2} = 0 \tag{5.8}$$

and

$$\frac{\varepsilon_1}{\varepsilon_2} = \alpha \tag{5.9}$$

Let

$$\Sigma_{as} = \begin{bmatrix} R_o & 0 \\ gQ_oK & \alpha^{-1}P_o \end{bmatrix} \tag{5.10}$$

and denote $\| P_o^{-1}Q_oKR_o^{-1} \| = a$, $\| B_o \| = b$ and $\| A_o \| = c$, then if

$$g \leq \frac{\sqrt{1 + \frac{4}{\varepsilon_1 c}\alpha(\varepsilon_1+b)} - 1}{2\alpha ab\varepsilon_1(1+\varepsilon_1)} \equiv g_c \tag{5.11}$$

then the solution of system (5.1),(5.3)-(5.5) can be approximated by

$$\begin{bmatrix} \chi(t) \\ u(t) \end{bmatrix} = e^{\Sigma_{as}\frac{t}{\varepsilon_1}} \begin{bmatrix} \chi(0) \\ u(0) \end{bmatrix} + e^{\Sigma_{as}\frac{t}{\varepsilon_1}} \begin{bmatrix} R_o^{-1}S_o \\ -g\alpha P_o^{-1}Q_oKR_o^{-1}S_o \end{bmatrix} x(0)$$
$$+ \begin{bmatrix} -R_o^{-1}S_o \\ g\alpha P_o^{-1}Q_oKR_o^{-1}S_o \end{bmatrix} x(t) + 0(\varepsilon_1) \tag{5.12}$$

and

$$x(t) = e^{[A_o+g\alpha B_oP_o^{-1}Q_oKR_o^{-1}S_o]}x(0) + 0(\varepsilon_1) \tag{5.13}$$

<u>Proof</u>: Condition (5.8) allows us to consider system (5.1), (5.3)-(5.5) as a singularly perturbed system in the form of (C 9), (C 10) with parasitic parameter ε_1. Thus, the block triangularization procedure in [24,25] is

applicable. The remainder of the proof follows that of Theorem 2.1 which is given in the Appendix B.

Corollary 5.1:

Suppose

$$-P_o^{-1}Q_o = I_m, \quad -R_o^{-1}S_o = I_n, \tag{5.14}$$

and let $t_{as} = |\max\lambda(\sum_{as})|^{-1}\varepsilon_1 \ell n \varepsilon_1$, then

$$\chi(t) = x(t) + 0(\varepsilon_1) \tag{5.15}$$

$$u(t) = g\alpha Kx(t) + 0(\varepsilon_1) \tag{5.16}$$

for $t \geq t_{as}$ and

$$x(t) = e^{[A_o + g\alpha B_o K]t} x(0) + 0(\varepsilon_1). \tag{5.17}$$

Thus, if $\varepsilon_1 = \varepsilon_2$, that is, $\alpha = 1$ and the sensor and actuator systems are such that (5.14) is satisfied, then the control u(t) of (5.5) is $0(\varepsilon_1)$ close to the high gain feedback (5.2) after an initial transient period of $0(\varepsilon_1)$. Moreover, the motion x(t) of the high gain feedback system with sensor and actuator dynamics (5.1),(5.3)-(5.5) is $0(\varepsilon_1)$ close to the motion x(t) of (5.1),(5.2). If $\alpha \neq 1$, the approximations still hold but with the gain factor g multiplied by α. In Theorem 5.1, an upper bound for the gain g is specified for which the approximations (5.12),(5.13) hold. In the case when (5.14) does not hold, the next theorem examines the stability of the motion x(t).

Theorem 5.2:

Let T be the transformation as in (2.17) and let

$\tilde{x} = [\tilde{x}_1 \; \tilde{x}_2]^T \equiv Tx$. Let $\Gamma \equiv \alpha P_o^{-1} Q_o K R_o^{-1} S_o B_o$ and suppose (5.18)

$$\mathrm{Re}\lambda(\Gamma) < 0 \tag{5.19}$$

then under the same conditions in Theorem 5.1 the motion x(t) of (5.13) can be approximated by

$$\tilde{x}_1(t) = e^{Zt}\tilde{x}_1(0) + O(g^{-1}) + O(\varepsilon_1) \tag{5.20}$$

and

$$\tilde{x}_2(t) = e^{g\Gamma t}\tilde{x}_2(0) + O(g^{-1}) + O(\varepsilon_1) \tag{5.21}$$

where $\lambda(Z)$ are the transmission zeros of the triple $(KR_o^{-1}S_o, A_o, B_o)$. Furthermore, if g is sufficiently large and $\mathrm{Re}\lambda(Z) < 0$, then $\lim_{t\to\infty} \tilde{x}_1(t) = 0$ and $\lim_{t\to\infty} \tilde{x}_2(t) = 0$.

Proof: An $O(\varepsilon_1)$ approximation to x(t) in (5.13) is the solution of the system

$$\dot{x}_a = (A_o + g\alpha B_o P_o^{-1} Q_o K R_o^{-1} S_o)x_a, \quad x_a(0) = x(0). \tag{5.22}$$

Applying the transformation T to (5.22), we obtain

$$\dot{x}_{a_1} = (A_{11}^o - A_{12}^o K_{2a}^{-1} K_{1a})x_{a_1} + A_{12}^o K_{2a}^{-1} x_{a_2} \tag{5.23}$$

$$\mu\dot{x}_{a_2} = \mu\bar{H}_1 x_{a_1} + (\mu\bar{H}_2 + \Gamma)x_{a_2} \tag{5.24}$$

where $$[x_{a_1}\ x_{a_2}]^T \equiv Tx_a,$$

$$KR_o^{-1}S_oM^{-1} = [K_{1a}\ K_{2a}] \tag{5.25}$$

and K_{2a} is an $m \times m$ nonsingular matrix since by assumption (5.18) $K_{2a}B_2^o$ is nonsingular. Following Theorem 2.1, and letting

$$Z = A_{11}^o - A_{12}^oK_{2a}^{-1}K_{1a} \tag{5.26}$$

we have

$$x_{a_1}(t) = e^{Zt}\tilde{x}_1(0) + 0(g^{-1}) \tag{5.27}$$

$$x_{a_2}(t) = e^{g\Gamma t}\tilde{x}_2(0) + 0(g^{-1}) \tag{5.28}$$

By Theorem 2.2, $\lambda(Z)$ are the transmission zeros of the triple $(KR_o^{-1}S_o,A_o,B_o)$. This concludes the proof.

Corollary 5.2:

If $R_o^{-1}S_o = I_n$, then matrix Z of (5.26) coincides with matrix F_{11} of (2.28). If in addition $P_o^{-1}Q_o = I_m$, then the 0(g) eigenvalues of (5.1),(5.3)-(5.5) are the 0(g) eigenvalues of (5.1),(5.2) multiplying by the factor α.

Theorem 5.2 shows that the fast transients in high gain feedback system are influenced by the presence of sensor and actuator dynamics. Neverthelsss, if (5.19) is satisfied, then the fast transients decay exponentially in the time scale gt to the neighborhood of Kx = 0. The presnce of sensor dynamics alone alters the finite eigenvalues of the high

gain feedback system thus it affects the dynamical characteristics of $x(t)$ after the fast transient diminishes.

If there are disturbances and parameter variations in (5.1), then from the above theorems, we see that high gain feedback (5.3) rejects the same class of disturbances and parameter variations as when (5.2) is used, that is when there are no sensor and actuator dynamics. Thus, if the gain factor g is not "exceedingly large", for example, if $g \leq g_c$ as in (5.11), and the sensor and actuator dynamics do not destroy the stability of the system in the sense of Theorem 5.2, then we can proceed to design high gain feedback by assuming that the sensors and actuators do not exist.

We note that if (5.8) is satisfied, it is immaterial that ε_1 and ε_2 differ in orders of magnitudes, our conclusion made above is still valid. The results for case when $g = \beta_1\varepsilon_1$ and $g = \beta_2\varepsilon_2$ where β_1 and β_2 are finite positive constants will not be presented in this study. Preliminary investigations show that the interaction of the fast modes due to the use of high gain feedback and the natural fast modes in the sensor and actuator gives rise to high frequency oscillation.

Finally, we remark that the effect of actuator and sensor dynamics on VSS with sliding mode is that nonideal sliding mode occurs, that is, after some time t_1, the trajectories of the system are confined to some Δ-neighborhood of the switching planes $s(x) = 0$ as in (4.39). Theorem 4.1 shows that nonideal sliding mode is close to the ideal sliding mode, that is, when there are no sensor and actuator dynamics, in the sense of (4.40).

While sliding mode is robust with respect to sensor and actuator dynamics, the design of switching gains in the variable structure control has to take into account the matrices P_o, Q_o, R_o and S_o in (5.4), (5.5) such that the trajectories reach $s(x) = 0$. Let υ be a variable structure control

$$\upsilon = \Psi^o \chi \tag{5.29}$$

where Ψ^o denote the switching gain matrix in (4.19). Due to the actuator dynamics (5.5), u is a continuous function and therefore sliding mode does not occur on $s(x) = 0$. The trajectories $x(t)$ of the discontinuous feedback system (5.1), (5.4), (5.5), (5.29) are continuous. Thus, the switching of the gains Ψ^o of (5.29) is regular. Since the feedback gains in Ψ^o are bounded, in the region of the state space of x where Ψ^o is a fixed gain matrix, if we denote, $\Psi^o = K$ and let $g = 1$, Theorem 5.1 can be used to show

$$u(t) = \alpha P_o^{-1} Q_o K R_o^{-1} S_o x(t) + 0(\varepsilon_1) \tag{5.30}$$

$$\chi(t) = -R_o^{-1} S_o x(t) + 0(\varepsilon_1) \tag{5.31}$$

After some time t_{as} which is of $0(\varepsilon_1)$. If (5.14) is satisfied, then

$$u(t) = \alpha K x(t) = 0(\varepsilon_1) \tag{5.32}$$

and $\chi(t)$ as in (5.15). If Ψ^o is designed assuming that there are no sensor and actuator dynamics, then if $\alpha > 1$ and $|x(t)| \geq c_1 \varepsilon_1$ for some positive number c_1, the trajectories move towards $s(x) = 0$. When (5.14) does not hold, ignoring the presence of sensor and actuator dynamics may lead to a design of Ψ^o which does not guarantee $s(x) = 0$ be reached and hence nonideal sliding mode would not exist.

As we have mentioned in Section 4.2 that in the vicinity of the switching planes, the variable structure control υ in (5.29) can be approximated by $\upsilon = gK\chi$ as in (5.3) where $K\chi = 0$ approximates the switching planes $Kx = 0$. If $Kx = 0$ is reached then the trajectories of the VSS are in some neighborhood of $Kx = 0$, that is, nonideal sliding mode occurs. We note that the reaching of $Kx = 0$ depends only on α, the ratio between ε_1 and ε_2. If the nonidealities in the switching devices can be approximated by (5.3), then in the vicinity of $Kx = 0$, nonideal sliding mode is the motion of the high gain feedback system (5.1),(5.3)-(5.5). The major difference between VSS in sliding mode and high gain feedback systems with sensor and actuator dynamics is their motions from the initial states to the vicinity of $Kx = 0$. In VSS, reaching of $Kx = 0$ guarantees the existence of nonideal sliding mode even in the case when $\upsilon = \Psi^o\chi$ has the representation $\upsilon = gK\chi$ near $Kx = 0$. In contrast, reaching of $Kx = 0$ in high gain systems depends on the stability of the "fast" transient which is determined by the relative magnitudes of ε_1, ε_2 and g^{-1}. Thus we conclude that VSS in sliding mode is more robust than high gain feedback systems in the presence of sensor and actuator dynamics which are neglected in the design stages.

5.2 Decentralized Control

High gain feedback and variable structure control can be applied to decouple interconnected systems. For simplicity, we shall limit our consideration to systems with only two interconnected subsystems. Let S_1, S_2 be the two subsystems given by

$$S_1: \quad \dot{x}^1 = A^1x^1 + B^1u^1 + E_2^1x^2 \tag{5.33}$$

$$S_2: \quad \dot{x}^2 = A^2x^2 + B^2u^2 + E_1^2x^1 \tag{5.34}$$

where x^i, an n_i-vector and u^i, and m_i-vector are the states and control of the subsystem S_i, $i = 1,2$, respectively. Suppose that associated with S^i, there is a measurement vector y_i, an r_i-vector, of some of the "local" states,

$$y^1 = C^1x^1 \tag{5.35}$$

$$y^2 = C^2x^2 \tag{5.36}$$

and the objective is to regulate some "local" variables, by "local" feedback, that is, the variables

$$z^1 = D^1x^1, \tag{5.37}$$

$$z^2 = D^2x^2 \tag{5.38}$$

are to be regulated in the subsystem S_1 and S_2, respectively, and the feedback controls are of the form

$$u^1 = h^1(y^1), \tag{5.39}$$

$$u^2 = h^2(y^2) \tag{5.40}$$

where h^i denote some unspecified feedback function. We note that the structure of the interconnected system given by (5.33), (5.34) are less general than a linear multivariable system with an $(n_1 + n_2)$-dimensional

state space and $(m_1 + m_2)$ control variables, namely, the control variables u^2 do not appear in S_1 and vice versa. The regulation problem we pose for S_1 and S_2 is a typical one in a decentralized control framework: each subsystem has access to measurements of its own states and its controller is designed based on the knowledge of its own dynamics. We recognize that if the interconnections $E^i_j x^j$ for S_i are considered as unmeasurable disturbances, then using high gain feedback control or variable structure control, it is possible to achieve regulation of z^i. Our approach is to construct "disturbance rejecting" observers for each subsystem using the results in Section 3.2 and 4.4, then local high gain observer feedback are designed accordingly. For ease of illustration, we first assume all the subsystem states can be measured locally, that is, $C^1 = I_{n_1}$ and $C^2 = I_{n_2}$ and the system matrices A^i_j, B^i and E^i_j are known and fixed at their nominal values.

Lemma 5.1:

Suppose local high gain feedback

$$u^i = g_i K^i x^i, \quad i = 1,2 \tag{5.41}$$

are applied to (5.33), (5.34) and suppose there exists a finite positive constant $\alpha = \frac{g_2}{g_1}$ and matrix K^i is designed such that the $O(g_i)$ eigenvalues of the high gain system (5.33), (5.41) and the transmission zeros of the triple (K^i, A^i, B^i) are placed at prescribed locations. Then there exists a finite positive number t_1 such that $z^1(t)$ and $z^2(t)$ are reduced to $O(g_1^{-1})$ quantities for all $t \geq t_1$, that is, $|z^i(t)| = O(g_1^{-1})$, $i = 1,2$ if and

only if the transmission zeros of $\left(\begin{bmatrix} K^1 & 0 \\ 0 & K^2 \end{bmatrix}, \begin{bmatrix} A^1 & E_2^1 \\ E_1^2 & A^2 \end{bmatrix}, \begin{bmatrix} B^1 & 0 \\ 0 & B^2 \end{bmatrix}\right)$ all lie in the open left half complex plane.

Proof: By Theorem 2.1 and 2.2, the finite eigenvalues of the composite high gain feedback system (5.33), (5.34), (5.41) are the transmission zeros of the composite system (5.33), (5.34) with "output" matrix $\begin{bmatrix} K^1 & 0 \\ 0 & K^2 \end{bmatrix}$. Since the $0(g_1)$ eigenvalues of this system are the union of the $0(g_1)$ eigenvalues of the high gain systems (5.33), (5.41) and (5.34), (5.41), the regulation of $z^1(t)$ and $z^2(t)$ results from the asymptotic stability of the slow motion in the composite system.

In order to guarantee that the transmission zeros of the composite system to lie in the open left half complex plane, the matrices K^1 and K^2 have to be designed "centrally," that is for the system (5.33), (5.34),

$$\dot{x}_c = A_c x_c + B_c u_c \tag{5.42}$$

where $A_c = \begin{bmatrix} A_2^1 & E_2^1 \\ E_1 & A^2 \end{bmatrix}$ and $B_c = \begin{bmatrix} B^1 & 0 \\ 0 & B^2 \end{bmatrix}$, we design a high gain feedback control of the form

$$u_c = g_1 \begin{bmatrix} K^1 & 0 \\ 0 & \alpha K^2 \end{bmatrix} x_c \equiv g_1 K_c x_c \tag{5.43}$$

such that for g_1 sufficiently large, the finite eigenvalues of (5.42), (5.43) have negative real parts.

Lemma 5.2

Let $\overline{K} = \{K \mid K = \text{block diag } (K_1, K_2)\}$ and let the set of fixed modes of (A_c, B_c) with respect to $\overline{K}$ be defined as

$$\Lambda(A_c, B_c, \overline{K}) = \bigcap_{K \in \overline{K}} \lambda(A_c + B_c K).$$

Then there exist matrices K^1 and K^2 such that the transmission zeros of

$\left(\begin{bmatrix} K^1 & 0 \\ 0 & K^2 \end{bmatrix}, A_c, B_c\right)$ lie in the open left half complex plane if and only if $\Lambda(A_c, B_c, \overline{K})$ lie in the open left half complex plane.

Proof: The matrix $g_1 K_c$ of (5.43) belongs to the set $\overline{K}$. If there exists a complex number $\sigma \in \Lambda(A_c, B_c, \overline{K})$, then by Theorem 2.2 σ is a transmission zero of (block diag (K_1, K_2), A_c, B_c) since the $0(g_1)$ eigenvalues of (5.42), (5.43) can be placed at prescribed locations. This proves the lemma.

An algorithm to find fixed modes is given in [49]. By applying the techniques in Section 2.1 for separating the slow and fast modes of high gain feedback systems, it can be shown that the transmission zeros of the composite systems are the eigenvalues of the matrix

$$\begin{bmatrix} A^1_{11} & E^1_{211} \\ E^2_{111} & A^2_{11} \end{bmatrix} - \begin{bmatrix} A^1_{12} & E^1_{212} \\ E^2_{112} & A^2_{12} \end{bmatrix} \begin{bmatrix} (K^1_2)^{-1} K^1_1 & 0 \\ 0 & (K^2_2)^{-1} K^2_1 \end{bmatrix} \tag{5.44}$$

where

$$M^i A^i (M^i)^{-1} = \begin{bmatrix} A^i_{11} & A^i_{12} \\ A^i_{21} & A^i_{22} \end{bmatrix}, \quad M^i B^i = \begin{bmatrix} 0 \\ B^i_2 \end{bmatrix} \tag{5.45}$$

$$M^i E^i_j = \begin{bmatrix} E^i_{j1} \\ E^i_{j2} \end{bmatrix}, \quad i \neq j \tag{5.46}$$

$$E^i_{jk} (M^j)^{-1} = [E^i_{jk1} \quad E^i_{jk2}], \quad k = 1,2, \; i \neq j \tag{5.47}$$

$$K^i (M^i)^{-1} = [K^i_1 \quad K^i_2] \tag{5.48}$$

for $i,j = 1,2$. Due to the structure of the matrix in (5.44), algorithm such as the one given in [49] for finding decentralized stabilizing control

can be used. Although we manage to find a high gain decentralized control, the design of it requires the knowledge of the whole composite system. In this design, we have not taken advantage of high gain feedback; namely, the insensitivity property of the slow motion with respect to disturbances. In the following theorem, this property is exploited to decentralize the design process.

Theorem 5.3:

Under the same assumptions of Lemma 5.1, if either

$$R(E_2^1) \subseteq R(B^1) \tag{5.49}$$

or

$$R(E_1^2) \subseteq R(B^2) \tag{5.50}$$

then there exists a finite positive number t_1 such that $|z^1(t)| = 0(g_1^{-1})$ and $|z^2(t)| = 0(g_1^{-1})$ for all $t \geq t_1$. Moreover, the transmission zeros of $\begin{matrix} K^1 & 0 \\ 0 & K^2 \end{matrix}$, A_c, B_c are the union of the set of transmission zeros of (K^1, A^1, B^1) and (K^2, A^2, B^2).

Proof: If (5.49) holds, then $E_{211}^1 = 0$ and $E_{212}^1 = 0$ in (5.44). The transmission zeros of the composite systems are the eigenvalues of the matrix

$$\begin{bmatrix} A_{11}^1 - A_{12}^1(K_2^1)^{-1}K_1^1 & 0 \\ E_{111}^2 - E_{112}^2(K_2^1)^{-1}K_1^1 & A_{11}^2 - A_{12}^2(K_2^2)^{-1}K_1^2 \end{bmatrix} \tag{5.51}$$

If (5.50) holds, $E_{111}^2 = 0$, $E_{112}^2 = 0$ and (5.44) becomes

$$\begin{bmatrix} A_{11}^1 - A_{12}^1(K_2^1)^{-1}K_1^1 & E_{211}^1 - E_{212}^1(K_2^2)^{-1}K_1^2 \\ 0 & A_{11}^2 - A_{12}^2(K_2^2)^{-1}K_1^2 \end{bmatrix} \tag{5.52}$$

In both cases, the transmission zeros are the eigenvalues of the matrices on the main diagonal which are the transmission zeros of (K^1,A^1,B^1) and (K^2,A^2,B^2). Since they are placed separately by K^1 and K^2 to prescribed location on the open left half complex plane, regulation of $z^1(t)$ and $z^2(t)$ is achieved.

Thus, if either (5.49) or (5.50) holds, then the design of the high gain decentralized control (5.43) is decomposed into two design problems: each subsystem solves for its own high gain feedback control (5.41) knowing only its own dynamics. The conditions (5.49), (5.50) mean that the slow subsystems of S_1 and S_2 are connected in cascades. For interconnected systems with more than two subsystems, if the slow subsystems of its subsystems have a multi-level structure [50], that is, the "A" matrix in the "slow"composite system has upper or lower triangular form, then the design process of high gain feedback (5.43) is decentralized. This is summarized in the next Theorem.

Theorem 5.4:

Suppose there are κ subsystems, $S_j, j = 1,\ldots\kappa$. Under the assumptions of Lemma 5.1, if there exists a set of integers $(i_1,\ldots,i_{\kappa-1})$ such that

$$R(E_k^{i_\sigma}) \subseteq R(B^{i_\sigma}) \tag{5.53}$$

for $k \notin \{i_1,\ldots,i_{\sigma-1}\}$, $\sigma = 1,\ldots,\kappa - 1$ with $i_o \equiv \emptyset$, then there exists a finite positive number t_1 such that $|z^j(t)| = 0(g_1^{-1})$, $j = 1,\ldots,\kappa$ for $t \geq t_1$. Moreover the transmission zeros of (block diag $(K^1,\ldots,K^\kappa)$, A_c, block diag $(B^1,\ldots,B^\kappa)$) are the union of the sets of transmission zeros of (K^j,A^j,B^j), $j = 1,\ldots,\kappa$ where A_c and E_k^i are defined analogously as in (5.42).

We now consider the problem of designing high gain observers for the subsystems. Again for simplicity, we deal with the system (5.34)-(5.36). For subsystem S_i, given the local measurements $y^i(t)$, we construct an observer to estimate $x^i(t)$. Let the systems

$$\dot{\hat{x}}^i = (A^i - \hat{g}_i H^i C^i)\hat{x}^i + B^i u^i \tag{5.54}$$

represent the decentralized high gain observers where H^1 and H^2 are the design matrices. We assume that the pairs (A^i,C^i) are detectable. Suppose high gain observer feedback

$$u^i = g_i K^i \hat{x}^i \tag{5.55}$$

is applied to (5.33),(5.34).

<u>Theorem 5.5</u>:

Suppose that matrix K^i is designed such that $\lambda(K^iB^i)$ and the transmission zeros of (K^i,A^i,B^i) are placed at prescribed locations. The matrix H^i is designed such that $\lambda(C^iH^i)$ and the transmission zeros of (C^i,A^i,H^i) are placed at prescribed locations. Then

$$\lim_{t\to\infty} x^i(t) = 0 \text{ and } \lim_{t\to\infty} (x^i(t) - \hat{x}^i(t)) = 0,\ i = 1,2 \tag{5.56}$$

if and only if the set of fixed modes $\Lambda(A_c, B_c, \bar{K})$ defined in Lemma 5.2 lie in the open left half complex plane.

Proof: Using the results in Section 2.1, it can be shown that the large eigenvalues are proportional to $\lambda(K^iB^i)$ and $\lambda(C^iH^i)$ for sufficiently large gain. Moreover, the transmission zeros of the triple

$$\left(\begin{bmatrix} C^1 & 0 & 0 & 0 \\ -K^1 & K^1 & 0 & 0 \\ 0 & 0 & C^2 & 0 \\ 0 & 0 & -K^2 & K^2 \end{bmatrix}, \begin{bmatrix} A^1 & 0 & 0 & E_2^1 \\ 0 & A_2^1 & 0 & E_2^1 \\ 0 & E_1^2 & A^2 & 0 \\ 0 & E_1^2 & 0 & A^2 \end{bmatrix}, \begin{bmatrix} H^1 & & & \\ & B^1 & & 0 \\ & & H^2 & \\ 0 & & & B^2 \end{bmatrix}\right) \tag{5.57}$$

are the finite eigenvalues of the decentralized high gain feedback system (5.33)-(5.36), (5.54), (5.55) which are the eigenvalues of the matrix

$$\left[\begin{array}{cc|cc} \hat{A}_{11}^1 - \hat{A}_{12}^1(C_2^1)^{-1}C_1^1 & 0 & 0 & \hat{E}_{211}^1 - \hat{E}_{212}^1(K_2^2)^{-1}K_1^2 \\ 0 & \hat{A}_{11}^2 - \hat{A}_{12}^2(C_2^2)^{-1}C_1^2 & \hat{E}_{211}^1 - \hat{E}_{212}^1(K_2^2)^{-1}K_1^2 & 0 \\ \hline \multicolumn{2}{c|}{0} & A_{11}^1 - A_{12}^1(K_2^1)^{-1}K_1^1 & E_{112}^1 - E_{212}^1(K_2^2)^{-1}K_1^2 \\ & & E_{111}^2 - E_{112}^2(K_2^1)^{-1}K_1^1 & A_{11}^2 - A_{12}^2(K_2^2)^{-1}K_1^2 \end{array}\right] \tag{5.58}$$

where

$$\hat{M}^iA^i(\hat{M}^i)^{-1} = \begin{bmatrix} \hat{A}_{11}^i & \hat{A}_{12}^i \\ \hat{A}_{21}^i & \hat{A}_{22}^i \end{bmatrix}, \quad \hat{M}^iH^i = \begin{bmatrix} 0 \\ H_2^i \end{bmatrix} \tag{5.59}$$

$$\hat{M}^i E^i_j = \begin{bmatrix} \hat{E}^i_{j1} \\ \hat{E}^i_{j2} \end{bmatrix}, \quad i \neq j \tag{5.60}$$

$$\hat{E}^i_{jk}(\hat{M}^j)^{-1} = [\hat{E}^i_{jk1} \quad \hat{E}^i_{jk2}], \quad k = 1,2, \quad i \neq j \tag{5.61}$$

$$C^i(\hat{M}^i)^{-1} = [C^i_1 \quad C^i_2] \tag{5.62}$$

for i = 1,2. Thus, the transmission zeros are the eigenvalues of the matrices $\hat{A}^1_{11} - \hat{A}^1_{12}(C^1_2)^{-1}C^1_1$, $\hat{A}^2_{11} - \hat{A}^2_{12}(C^2_2)^{-1}C^2_1$ and the lower block diagonal matrix in (5.58). The eigenvalues of the first two matrices are the transmission zeros of (C^1, A^1, H^1) and (C^2, A^2, H^2), respectively. The third matrix is the same as (5.44) hence by Lemma 5.2, the theorem is proved.

The implication of this theorem is that if there exists decentralized stabilizing high gain state feedback, then by substituting the local states by the states of the decentralized high gain observers for feedback, the closed loop dynamical behavior of the interconnected system is close to that when full state feedback is used. Thus the separation property in linear observed feedback system holds in this case. We note that, in general, the separation property does not hold when regular gain decentralized observer feedback is applied. We observe that as in full state feedback, the design K^1 and K^2 is centralized although the designs of H^1 and H^2 are compledely independent of each other. From the structure of the matrix in (5.58), we see that if either condition, (5.49), (5.50), holds then again we have a totally decentralized design process. For composite system with more than two subsystems, the condition (5.53) in Theorem 5.4 is applicable.

5.3 Design Examples--High Gain Feedback

The synthesis procedures of high gain feedback control developed in Section 3.6 are now illustrated in the design of control systems for a distillation column and a synchronous machine.

5.3.1 Distillation Column

Precise control of product quality in distillation columns is considered to be an important application of automatic control in the chemical and petroleum industries. For a detail description of the nature of and the difficulties in controlling distillation processes, the readers are referred to [51]. Briefly, distillation is used in many chemical processes for separating feed streams and for purification of final and intermediate product streams. In actuality, most columns handle multicomponent feeds but many can be approximated by binary mixtures [52]. We will consider only binary distillation columns. A schematic diagram is given in Figure 5.1. The distillation unit consists of a distillation column which has, in this case, eight plates (or trays), a condenser and a reboiler unit. A single liquid feed stream is fed onto the feed plate which is the no. 4 plate. The vapor is condensed in a condenser and flows into the reflux drum. Reflux is pumped back to the top plate which is the no. 1 plate of the column. At the base of the column, vapor boilup is generated in a reboiler. In [53], an approximate linearized mathematical model of this distillation unit for small deviations in compositions and pressure about their operating points is derived. The essential assumptions made in this model are that

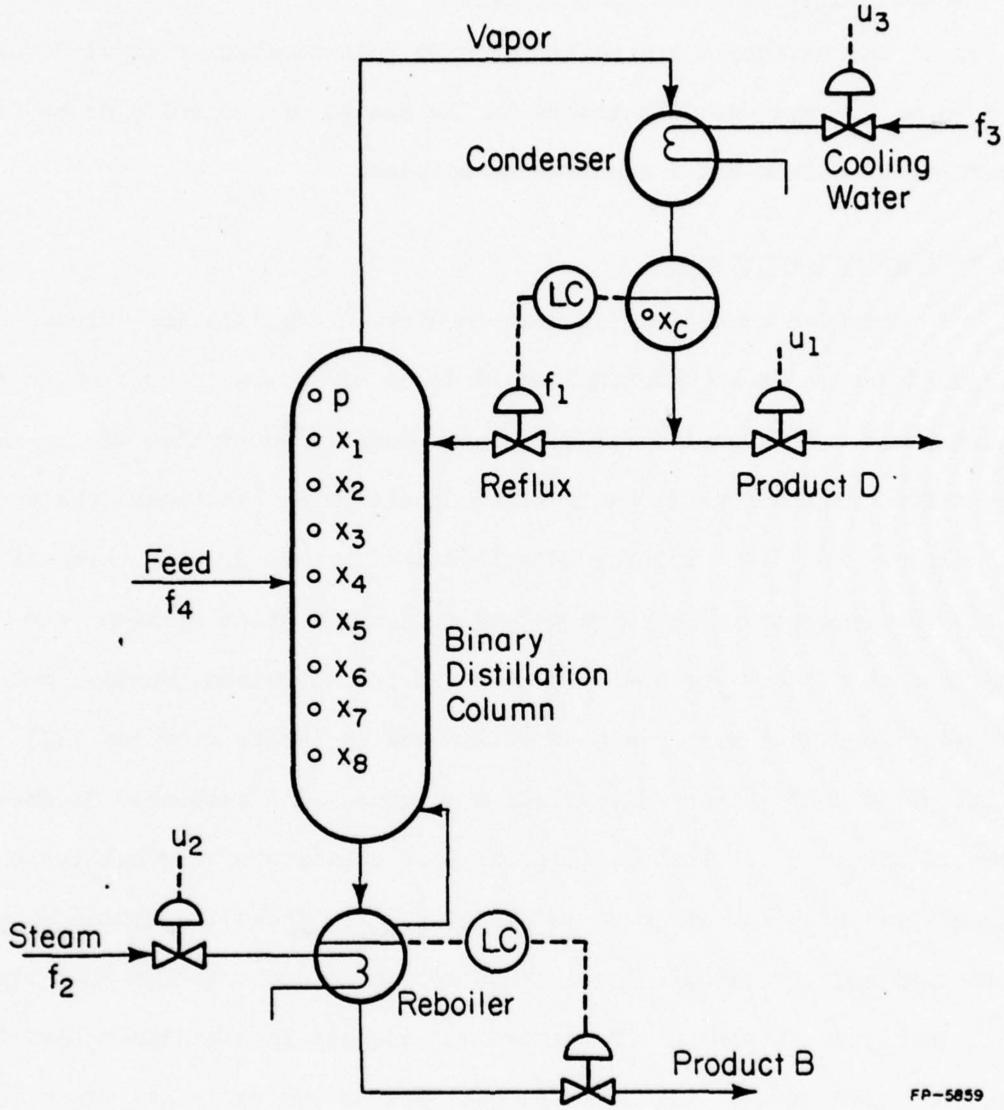


Figure 5.1 Schematic diagram of a binary distillation column, LC abbreviates for Liquid (level) Controller.

(i) the hydraulic delay occurring in the liquid flows is negligible.

(ii) the pressure variation is assumed constant throughout the whole column, implying that the pressure drops across the plates are negligible with respect to the total average change of pressure, and

(iii) the equation relating vapor composition to liquid composition on each plate is linearized about the operation point.

The model is described by a 11th order linear time-invariant system of the form $\dot{x} = A_o x + B_o u + E_o f$ with state vector $x^T = (x_c, x_1, \ldots, x_8, x_b, p)$ where x_i, $i = 1,\ldots,8$ are the liquid composition on i^{th} plate (mole fraction of more-volatile component); x_c and x_b are the liquid compositions in the condenser and reboiler, respectively and p is the pressure. The control variables are the reflux (u_1), the steam input temperature to reboiler (u_2) and the cooling water temperature to condenser (u_3). These three variables and the feed input are the disturbance inputs, $f_1 f_2, f_3$ and f_4 respectively. The matrices A_o, B_o and E_o are given in Table 5.1. For convenience, we have incorporated the feed flow FF and feed composition FCOM into a single disturbance input f_u, namely, $f_4 \equiv .018FF + .011FCOM$.

In a typical distillation control system, the steam and cooling water temperatures, u_2 and u_3, are the only control variables. There are two proportional loops, the steam temperature is proportional to the temperature near the bottom of the column and the coolant temperature is proportional to the pressure. In [53], a new feedback control structure is found by applying Rosenbrock's modal analysis procedure [54]. The objective of their design is to minimize the steady state error in output

$$A_o = \begin{bmatrix}
-.014 & .0043 & & & & & & & & & \\
.0095 & -.0138 & .0046 & & & & & & & & 5\text{x}10^{-4} \\
 & .0095 & -.0141 & .0063 & & & & & & & 2\text{x}10^{-4} \\
 & & .0095 & -.0158 & .011 & & & & & & \\
 & & & .0095 & -.0312 & -.015 & & & & & \\
 & & & & .0202 & -.0352 & .022 & & & & \\
 & & & & & .0202 & -.0422 & .028 & & & \\
 & & & & & & .0202 & -.0482 & .037 & & 2\text{x}10^{-4} \\
 & & & & & & & .0202 & -.0572 & .042 & 5\text{x}10^{-4} \\
 & & & & & & & & .0202 & -.0483 & 5\text{x}10^{-4} \\
 & & & & & & & & & .0255 & -.0185
\end{bmatrix}$$

$$B_o^T = \begin{bmatrix}
0 & .0025 & .005 & .005 & .005 & .005 & .005 & .055 & .0025 & .0025 & 0 \\
0 & .5\text{x}10^{-5} & .2\text{x}10^{-5} & .1\text{x}10^{-5} & 0 & 0 & -.5\text{x}10^{-5} & -10^{-5} & -4\text{x}10^{-5} & -2\text{x}10^{-5} & 4.6\text{x}10^{-4} \\
0 & -.4\text{x}10^{-5} & -2\text{x}10^{-5} & -10^{-5} & 0 & 0 & .1\text{x}10^{-5} & .3\text{x}10^{-5} & .5\text{x}10^{-5} & .2\text{x}10^{-5} & 4.6\text{x}10^{-4}
\end{bmatrix}$$

$$E_o^T = \begin{bmatrix}
0 & 0 & 0 & 0 & \overset{B_o^T}{1} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Table 5.1. Matrices A_o, B_o and E_o for the binary distillation unit. Only nonzero entries in A_o are shown.

product compositions, that is, x_c and x_b, caused by a constant disturbance, for example, the feed input. Thus, there are two variables to be regulated, that is, $z^T = [z_1 \; z_2]$ and $z_1 = x_c$ and $z_2 = x_b$. The matrix D_o where $z = D_o x$ is

$$D_o = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \qquad (5.63)$$

We note that the first row of D_o is orthogonal to the columns of B_o. Hence, $R(D_o^T) \subset N\,(B_o^T)$ which violates the necessary condition (3.112) for the existence of a matrix K that satisfies $R(D_o^T) \subseteq R(K^T)$. Alternatively let the temperatures of the condenser and reboiler T_c and T_b, be the regulated variables. These temperature variables are linear combinations of the pressure and the respective composition, that is,

$$D_o = \begin{bmatrix} 55 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 55 & 20 \end{bmatrix}. \qquad (5.64)$$

This matrix D_o satisfies (3.112). We verify that the pair $(A_{11}^{'o}, A_{12}^{'o}\Pi)$ in Theorem 3.13K is controllable. Therefore, there exists a high gain feedback control $u = gKx$ such that $z(t)$ is zeroed.

From Table 5.1, we see that the first three columns of B_o and E_o are identical. By Theorem 3.1, the sensitivity of $p(t)$ with respect to reflux, steam input and cooling water temperatures is inversely proportional to the gain factor g. Since $T_c(t)$ and $T_b(t)$ are being zeroed, the steady state errors in $x_c(t)$ and $x_b(t)$ (assuming that the feed input is constant) are proportional to the steady state error in $p(t)$ caused by the feed input. It is of interest to note that it is concluded in [53] that steady

state errors due to feed input cannot be eliminated which agrees with our observation.

In [53], it is suggested that the variables x_2, x_7 and p be measured. Since there are four disturbance inputs, Theorem 3.3 demands at least four measurements. If information on x_4, the composition on the feed plate (no. 4), is not contained in the measurements, that is, the fifth column of the measurement matrix C_o is a null vector, then the last column of E_o which is associated with the feed input lies in the null space of C_o. Thus in order to satisfy the necessary condition $\mathcal{R}(E_o) \cap \mathcal{N}(C_o) = \emptyset$ in Theorem 3.13H, we choose

$$C_o = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.65}$$

that is, $y^T = [y_1, y_2, y_3, y_4]$ and $y_1 = x_2$, $y_2 = x_4$, $y_3 = x_7$, $y_4 = p$. To check the existence of matrix H in the high gain observer (3.34), (3.35) which satisfies the specifications in Problem H, we proceed according to Theorem 3.13H. Since the number of measurements and number of disturbance inputs are the same, that is, $r = p = 4$ the structural constraint in Problem H becomes $\mathcal{R}(E_o) = \mathcal{R}(H)$ and by Corollary 3.9H, we need to check the transmission zeros of (C_o, A_o, E_o). We found that they are the real numbers {-.0047, -.0075, -.0203, -.0220, 0.0, .0176, .0289}. There are three transmission zeros in the closed right half complex plane. Hence the resulting high gain observer which satisfies $\mathcal{R}(E_o) = \mathcal{R}(H)$ is unstable. If we consider only the disturbances f_2, f_3 and f_4, then $r > p$. According to

Theorem 3.13H, we need to check the controllability of $(\overline{A}^o_{11}, \overline{A}^o_{12}\Pi)$. We found that it is controllable, hence the eigenvalues of the slow subsystem of the high gain observer can be placed at prescribed locations. However, there is steady state observation error due to disturbance f_1, the reflux, (assuming that it is constant). If we also measure x_5, that is $r = 5$, y_1 through y_4 are as before and $y_5 = x_5$, then it is possible to reject all disturbances. We found that the pair $(\overline{A}^o_{11}, \overline{A}^o_{12}\overline{\Pi})$ is stabilizable: there is an uncontrollable mode -.0198 in $\overline{A}^o_{11}$. With the exception of this mode, the eigenvalues of the slow subsystem can be placed and the observation errors are insensitive to all the disturbances.

5.3.2 <u>Synchronous Machine</u>

A linearized mathematical model of a salient pole synchornous machine connected to an infinite bus is given in [55]. The model is described by a 9th order linear time-invariant system of the form $\dot{x} = A_o x + B_o u + E_o f$ with two control variables and three disturbance variables. The state variables are the flux linkages of the field armature and damper windings (x_1, x_2, x_3, x_4), the terminal voltage (x_5), the field voltage (x_6), the speed of rotation (x_8) and the power angle (x_9). For a thermal machine, x_7 is the mechanical torque T_m. For a hydraulic machine, x_7 is defined to be $T_m + 2T_{ing}$ where T_{ing} is the gate position. The control variables are the reference signal to the exciter (u_1) and the gate position (u_2). These two variables and the disturbance torque to the machine shaft are the disturbance inputs, f_1, f_2 and f_3, respectively. The matrices A_o, B_o and E_o for a thermal machine are given in Table 5.2. For a hydraulic machine, the A_o matrix is the same as given in the table except for the following elements:

$$a_{44} = -26.3,\ a_{48} = -.0756,\ a_{49} = -4.87,\ a_{77} = -1.24. \tag{5.66}$$

The matrices B_o and E_o are the same as in a thermal machine except for the following elements:

$$b_{27} = 12.42, \quad b_{28} = -110. \tag{5.67}$$

We note that the ranges of the input matrix B_o for a thermal machine and a hydraulic machine are not the same. It is expected that their insensitivity properties with respect to the disturbances will be different. Since rank $E_o = 4 \geq$ rank $B_o = 2$, the maximum number of regulated variables is two. We select the field voltage and the speed of rotation to be the regulated variables, that is, $z_1 = x_6$, $z_2 = x_8$ and

$$D_o = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{5.68}$$

For a thermal machine, the second row of D_o is orthogonal to the columns of B_o and hence violates the necessary condition (3.112). If we design a high gain feedback $u = gKx$ ignoring the structural constraint $\mathcal{R}(D_o^T) \subseteq \mathcal{R}(K^T)$, then the steady state errors in $x_6(t)$ and $x_8(t)$ (assuming constant disturbances) are due to f_3, the shaft disturbance torque since the first two columns of E_o coincides with B_o. For a hydraulic machine, $\mathcal{R}(D_o^T) \cap \mathcal{N}(B_o^T) = \emptyset$. Since the number of regulated variables is the same as the number of control variables, that is, $\ell = m$, the structural constraint becomes $\mathcal{R}(D_o^T) = \mathcal{R}(K^T)$. By Corollary 3.9K, we need to check the transmission zeros of (D_o, A_o, B_o). We found that they are the real numbers $\{-100., -60.52, -34.03, -26.3, -19.16, .0011, 5.02\}$. There are two

$$A_o = \begin{bmatrix} -2.77 & -3.07 & 2.98 & & & 1 & & & -.599 \\ 26.8 & -61.5 & .524 & & & .176 & & -.0923 & -32.1 \\ 30 & -15.5 & -32.2 & & & & & & -15.6 \\ & & & -27.7 & & & & -.0828 & -5.32 \\ & 44 & & -89.8 & -100 & & & & \\ & & & & -3875 & -100.3 & & & \\ & & & & & & -3.33 & & \\ & -223 & & -47.8 & & & 55.4 & -.35 & -222 \\ & & & & & & & 1 & 0 \end{bmatrix}$$

$$B_o^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3.33 & 0 & 0 \end{bmatrix}$$

$$E_o^T = \begin{bmatrix} & & & & B_o^T & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Table 5.2. Matrices A_o, B_o and E_o for a thermal synchronous machine connected to an infinite bus. Only nonzero entries in A_o are shown.

transmission zeros in the right half complex plane. Hence the resulting high gain feedback system which satisfies $\mathcal{R}(D_o^T) = \mathcal{R}(K^T)$ is unstable. If we consider only the regulation of the field voltage, x_6, then $m > \ell$. According to Theorem 3.13K, we need to check the controllability of $(A_{11}^{'o}, A_{12}^{'o}\Pi)$. We found that it is controllable. Thus, the eigenvalues of the slow subsystem of the high gain feedback system can be placed and $x_6(t)$ is regulated. This is also the case for a thermal machine. On the other hand, if we consider only the regulation of the speed, x_8, then the corresponding pair $(A_{11}^{'o}, A_{12}^{'o}\Pi)$ has an uncontrollable mode 1.06×10^{-2}. Hence the resulting high gain feedback system which satisfies $\mathcal{R}(D_o^T) \subset \mathcal{R}(K^T)$ is unstable. If this structural constraint is replaced by the constraint as if $x_6(t)$ is to be regulated, then $x_6(t)$ is zeroed and the steady state error in $x_8(t)$ is due to f_3, the shaft disturbance torque only.

In [7], it is suggested that the variables x_5, x_6, x_8 and x_9 be measured, that is,

$$C_o = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.69}$$

$y^T = [y_1, y_2, y_3, y_4]$ and $y_1 = x_5$, $y_2 = x_6$, $y_3 = x_8$, $y_4 = x_9$. For a thermal machine, the second column of E_o lies in the null space of C_o. Hence $\mathcal{R}(E_o) \subset \mathcal{N}(C_o)$ which violates the necessary condition (3.142). Thus, there are steady state observation errors due to constant disturbances f_2, the gate position. If we consider only the first and third column of E_o, then $r > p$ and we found that the corresponding pair $(\overline{A}_{11}^{o}, \overline{A}_{12}^{o}\overline{\Pi})$ in Theorem

3.13H is stabilizable: there is an uncontrollable mode -1.24. With the exception of this mode, the eigenvalues of the slow subsystem can be placed and the observation errors are insensitive to f_1 and f_3, the exciter reference signal and the shaft disturbance torque. Since the first and third column of E_o in a hydraulic machine are identical to that of a thermal machine, these conclusions also hold. We note that in a hydraulic machine, the second column of E_o lie outside the null space of C_o but the span of the second and third column of E_o lie inside the null space of C_o. If we consider only the first two columns of E_o, we found that the pair $(\overline{A}^o_{11}, \overline{A}^o_{12}\overline{\Pi})$ has an uncontrollable mode 5.02. The resulting high gain observer is unstable. We conclude that in either a thermal or hydraulic machine, the observation error cannot be made insensitive with respect to f_2, the gate position disturbance.

5.4 Design Example--Variable Structure Feedback

We illustrate the synthesis procedures of variable structure feedback controllers in the design of a model following control system. The idea of model following is to use a model, which specifies the design objectives, as a part of the control system. The objective of the controller synthesis is then to minimize the errors between the states of the model and the plant (the system to be controlled). In the presence of system parameter variations, the so-called "adaptive model following control system" are developed. Basically, the gains of the controller in these systems are adjusted so that the plant states track the model states. Let the model be given by

$$\dot{x}_M = A_M x_M + B_M u_M \tag{5.70}$$

where $x_M \in R^n$ is the model state vector and $u_m \in R^K$ is the model input. Suppose the plant is described by

$$\dot{x}_p = A_p x_p + B_p u_p \tag{5.71}$$

with $x_p \in R^n$ and $u_p \in R^m$ are the state vector and control vector of the plant, respectively. Then the error between the model and plant states,

$$e = x_M - x_p \tag{5.72}$$

is governed by

$$\dot{e} = A_M e + (A_M - A_p)\, x_p + B_M u_M - B_p u_p. \tag{5.73}$$

In general, the matrices A_M and B_M are known but there are parameter variations in A_p and B_p. The objective is then to design a controller u_p such that the error $e(t)$ goes to zero asymptotically. Our approach is to design a variable structure control u_p considering x_p and u_M as "disturbances." Since x_p and u_M are accessible, we let

$$u_p = \Psi_p x_p + \Psi_e e + \Psi_M u_M. \tag{5.74}$$

The matrices Ψ_p, Ψ_e and Ψ_M are defined as follows.

$$\Psi_p = \tilde{\Psi}_p \operatorname{diag}(\operatorname{sgn} x_p^1, \ldots, \operatorname{sgn} x_p^n), \quad x_p^T = (x_p^1, \ldots, x_p^n) \tag{5.75}$$

$$\Psi_e = \tilde{\Psi}_e \operatorname{diag}(\operatorname{sgn} e_1, \ldots, \operatorname{sgn} e_n), \quad e^T = (e_1, \ldots, e_n) \tag{5.76}$$

$$\Psi_M = \tilde{\Psi}_M \operatorname{diag}(\operatorname{sgn} u_M^1, \ldots, \operatorname{sgn} u_M^{\kappa}), \quad u_M^T = (u_M^1, \ldots, u_M^{\kappa}) \tag{5.77}$$

where

$$\tilde{\Psi}_p^T = (\psi_p^1, \ldots, \psi_p^m), \; \psi_p^i = \begin{cases} k_p^i & \text{for } s_i(e) > 0 \\ -k_p^i & \text{for } s_i(e) < 0 \end{cases}, \; i = 1, \ldots, m \tag{5.78}$$

$$\tilde{\Psi}_e^T = (\psi_e^1, \ldots, \psi_e^m), \; \psi_e^i = \begin{cases} k_e^i & \text{for } s_i(e) > 0 \\ -k_e^i & \text{for } s_i(e) < 0 \end{cases}, \; i = 1, \ldots, m \tag{5.79}$$

$$\tilde{\Psi}_M^T = (\psi_M^1, \ldots, \psi_M^m), \; \psi_M^i = \begin{cases} k_M^i & \text{for } s_i(e) > 0 \\ -k_M^i & \text{for } s_i(e) < 0 \end{cases}, \; i = 1, \ldots, m \tag{5.80}$$

In the above equations, ψ_p^i, k_p^i, ψ_e^i and k_e^i *are* $n \times 1$ *vectors and* ψ_M^i, k_M^i are $\kappa \times 1$ vectors. The switching plane $s_i(e) = 0$ is the i^{th} component of

$$s(e) = Ge = 0. \tag{5.81}$$

By Lemma 4.5, if

$$\operatorname{rank}\,[B_p \,\vdots\, A_M - A_p] = \operatorname{rank} B_p \tag{5.82}$$

and

$$\operatorname{rank}\,[B_p \,\vdots\, B_M] = \operatorname{rank} B_p \tag{5.83}$$

then the VSS (5.73), (5.74) in sliding mode is described by

$$\dot{e} = [I - B_p(GB_p)^{-1}G]A_M e \tag{5.84}$$

which is insensitive to u_M and x_p. By designing matrix G such that the transmission zeros of (G, A_M, B_p) are placed at prescribed locations, shaping of the error transient after sliding mode occurs on $s(e) = 0$ is achieved.

We now design the variable structure control (5.74) for an aircraft control problem. The plant in this case represents the three degrees-of-freedom linearized longitudinal state equations of a conventional subsonic aircraft, a Convair C-131B. The model is chosen to be the estimated dynamics of a large supersonic aircraft. This is a benchmark problem considered in various model following papers [56,57,58]. The model and plant matrices are given as follows [57,58].

$$A_M = \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 \\ 5.318\times10^{-7} & -0.4179 & -0.1202 & 2.319\times10^{-3} \\ -4.619\times10^{-9} & 1.0 & -0.7523 & -2.387\times10^{-2} \\ -0.5614 & 0.0 & 0.3002 & -1.743\times10^{-2} \end{bmatrix} \tag{5.85}$$

$$B_M = \begin{bmatrix} 0.0 & 0.0 \\ -0.1717 & 7.451\times10^{-6} \\ -0.0238 & -7.783\times10^{-5} \\ 0.0 & 3.685\times10^{-3} \end{bmatrix} \tag{5.86}$$

$$A_p = \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 \\ 1.401\times10^{-4} & \sigma & -1.9513 & 0.0133 \\ -2.505\times10^{-4} & 1.0 & -1.3239 & -0.0238 \\ -0.561 & 0.0 & 0.358 & -0.0279 \end{bmatrix}, \tag{5.87}$$

$$B_p = \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ -5.3307 & 6.447\times10^{-3} & -0.2669 \\ -0.16 & -1.155\times10^{-2} & -0.2511 \\ 0.0 & 0.106 & 0.0862 \end{bmatrix}. \tag{5.88}$$

The plant control variables u_p^1, u_p^2 and u_p^3 are the elevator command deflection, throttle control and flap command deflection where u_M^1, u_M^2 are the elevator and throttle inputs, respectively. The plant state variables x_p^1, x_p^2, x_p^3 and x_p^4 are the pitch angle, pitch rate, angle of attack and air speed, respectively. The model state variables x_M^1, x_M^2, x_M^3 and x_M^4 denote similar quantities as the plant state variables. The plant matrices in (5.87), (5.88) contain only one parameter σ which varies between -.558 and -3.558. We see that conditions (5.82) and (5.83) are satisfied. By letting

$$G = \begin{bmatrix} \alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.89}$$

the equivalent control system which governs the dynamics of the VSS when sliding mode occurs on the intersections of the switching planes $s = Ge = 0$ is given by a first order system

$$\dot{e}_1 = -\alpha e_1 \tag{5.90}$$

where e_1 is the first element of the error vector e. We conclude that sliding mode is insensitive to the parameter σ and by choosing α to be positive, we guarantee the error goes to zero asymptotically. Furthermore, $-\alpha$ is in fact the eigenvalue of (5.90), thus the transient behavior is determined by the selection of the switching plane $s_1(e) = \alpha e_1 + e_2 = 0$. We choose $\alpha = 10$. The hierarchy of controls method is used to design the gains in the variable structure control law. We choose $s_2 \rightarrow s_3 \rightarrow s_1$

as the hierarchy of switching planes where

$$s_1(e) = 10e_1 + e_2 = 0 \tag{5.91}$$

$$s_2(e) = e_3 = 0 \tag{5.92}$$

$$s_3(e) = e_4 = 0. \tag{5.93}$$

To conform with the labeling convention of the hierarchy of controls made in Step 1 of the hierarchy of controls method, we relabel $s_1' = s_2$, $s_2' = s_3$ and $s_3' = s_1$ and let u_p^i be discontinuous on the switching plane $s_i'(e) = 0$, that is,

$$u_p^i = [(k_e^i)^T|e| + (k_p^i)^T|x_p| + (k_M^i)^T|u_M|)\operatorname{sgn} s_i', \quad i = 1,\ldots,3. \tag{5.94}$$

The following conditions on the gains k_e^i, k_p^i and k_M^i are obtained from the hierarchy of controls method.

$$\begin{aligned} (k_e^1)^T &\geq [2.08 \quad 12.82 \quad 11.49 \quad 1.80] \\ (k_e^2)^T &\geq [6.11 \quad 3.26 \quad 6.10 \quad .98] \\ (k_e^3)^T &\geq [\ .27 \quad 3.05 \quad 3.07 \quad .12] \end{aligned} \tag{5.95}$$

$$\begin{aligned} (k_p^1)^T &\geq [4\times10^{-3} \quad 1.65 \quad 8.50 \quad .23] \\ (k_p^2)^T &\geq [4.6\times10^{-3} \quad .82 \quad 2.99 \quad .19] \\ (k_p^e)^T &\geq [10^{-3} \quad .41 \quad 2.19 \quad .07] \end{aligned} \tag{5.96}$$

$$\begin{aligned} (k_M^1)^T &\geq [.32 \quad .03] \\ (k_M^2)^T &\geq [.09 \quad .05] \\ (k_M^3)^T &\geq [.08 \quad 1.5\times10^{-3}]. \end{aligned} \tag{5.97}$$

where $[a_1 \;\; b_1] \geq [a_2 \;\; b_2]$ means $a_1 \geq a_2$ and $b_1 \geq b_2$. If these conditions are satisfied, then sliding mode will occur on $s_2(e) = 0$, then on the intersection of $s_2(e) = 0$ and $s_3(e) = 0$ and finally on the intersections of all the switching planes. We let the variable structure control (5.94) be

$$\begin{bmatrix} u_p^1 \\ u_p^2 \\ u_p^e \end{bmatrix} = \mathrm{diag}(\mathrm{sgn}\, s_2, \mathrm{sgn}\, s_3, \mathrm{sgn}\, s_1)\left\{\begin{bmatrix} 3 & 13 & 12 & 2 \\ 7 & 4 & 7 & 1 \\ 1 & 4 & 4 & 1 \end{bmatrix}|e| + \right.$$

$$\left. + \begin{bmatrix} .01 & 2 & 9 & 1 \\ .01 & 1 & 3 & 1 \\ .001 & 1 & 3 & .1 \end{bmatrix}|x_p| + \begin{bmatrix} .1 & .1 \\ .1 & .1 \\ .1 & .01 \end{bmatrix}|u_M| \right\} \tag{5.98}$$

We have simulated the aircraft dynamics (5.87), (5.88) the reference model (5.85), (5.86) and the variable structure control law using the Runge-Kutta fourth order integration subroutine with automatic step size adjustments in the IBM Scientific Subroutine Package [59]. Double precision arithmetic is used and the error tolerance for numerical integration is 10^{-3}. The parameter σ in A_p is varied such that it is uniformly distributed between -0.558 and -3.558. The reference model inputs $u_M^1(t)$ and $u_M^2(t)$ are taken to be the same as in [9], that is,

$$u_M^1(t) = \begin{cases} 1 & 0 < t \leq 20 \text{ sec. and } 40 \text{ sec.} \leq t \leq 60 \text{ sec.} \\ 0 & 20 \text{ sec.} < t < 40 \text{ sec.} \end{cases} \tag{5.99}$$

The model trajectories which are to be tracked by the plant trajectories are illustrated in Figure 5.2.

The simulation results we have obtained indicate that sliding mode occurs as designed. Among these results, we focus on the result corresponding to the initial plant model errors $e_1(0) = -1$, $e_2(0) = e_3(0) = e_4(0) = 0$. The error trajectories are illustrated in Figure 5.3. In [57,58], error responses for the same initial errors using linear model following control and adaptive model following control are obtained. A comparison of their responses to those in Figure 5.3 shows that significant improvements on the transient behavior are made using the variable structure model following control. We note that sliding mode occurs on $s_2(e) = 0$ and $s_3(e) = 0$ instantaneously since $e_3(0) = e_4(0) = 0$. The trajectory in the phase plane (e_1,e_2) is illustrated in Figure 5.4. It is found that sliding mode occurs on the intersections of all the switching planes at $t \approx .3$ sec. The time constant of the trajectory $e_1(t)$ for $t \geq .3$ sec. is approximately .1 which confirms that sliding mode is indeed described by (5.90) with $\alpha = 10$.

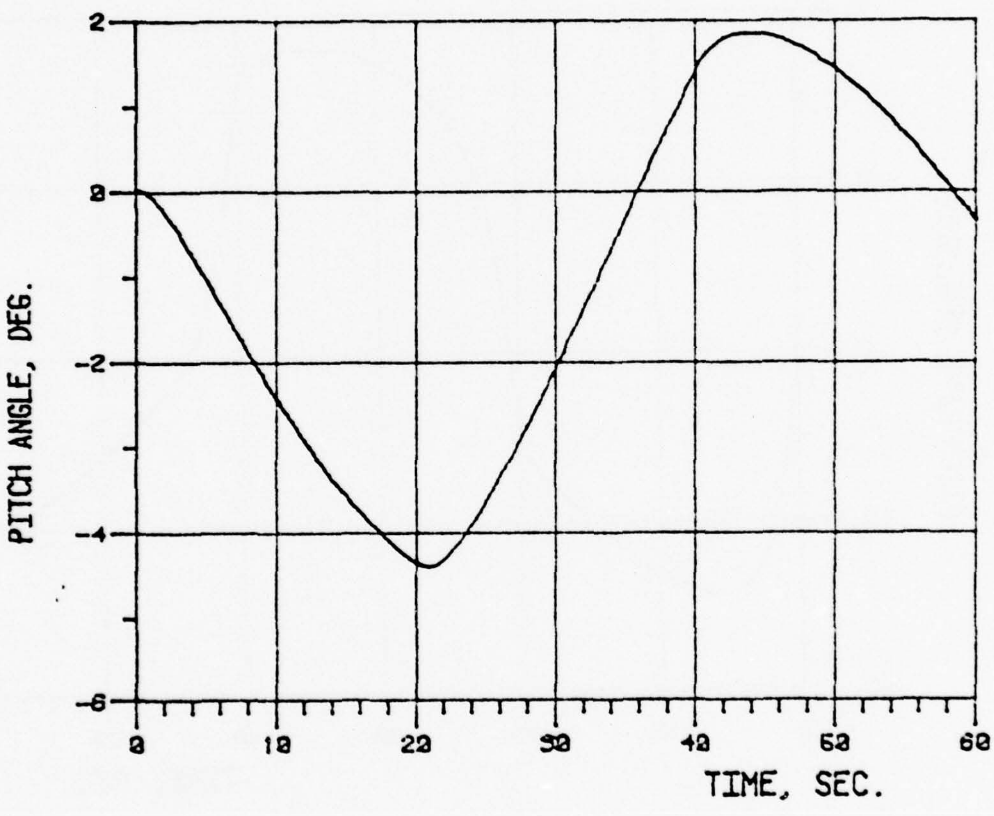


Figure 5.2a Pitch angle trajectory of the model: $x_M^1(t)$.

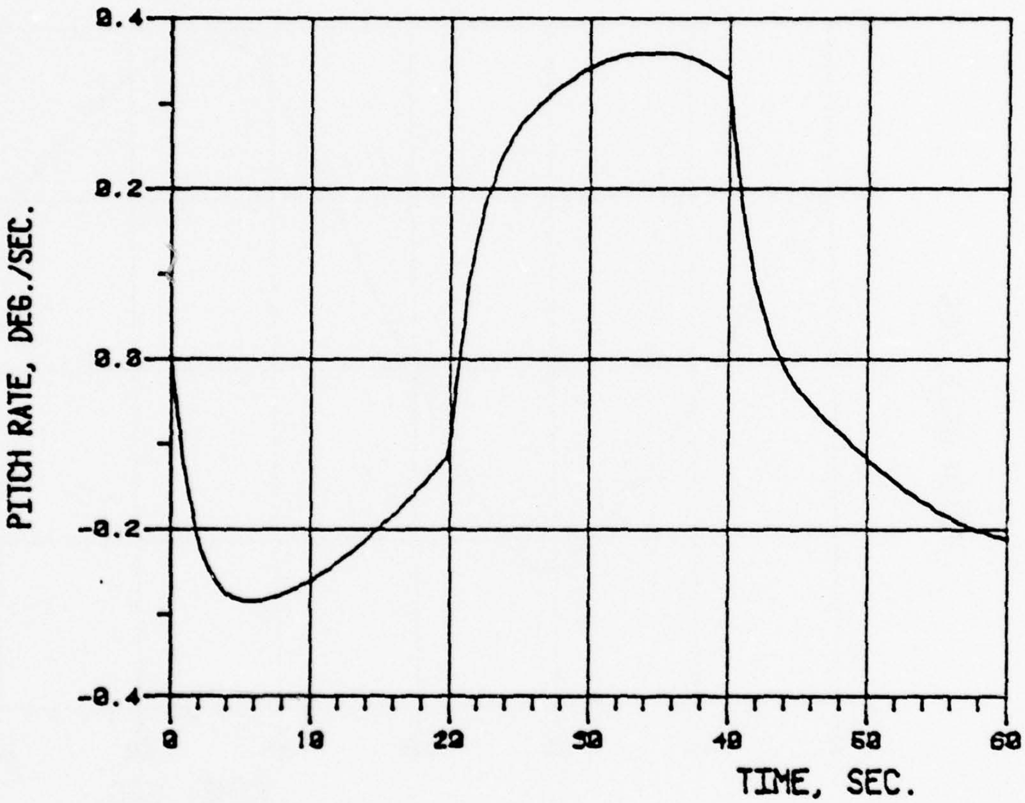


Figure 5.2b Pitch rate trajectory of the model: $x_M^2(t)$.

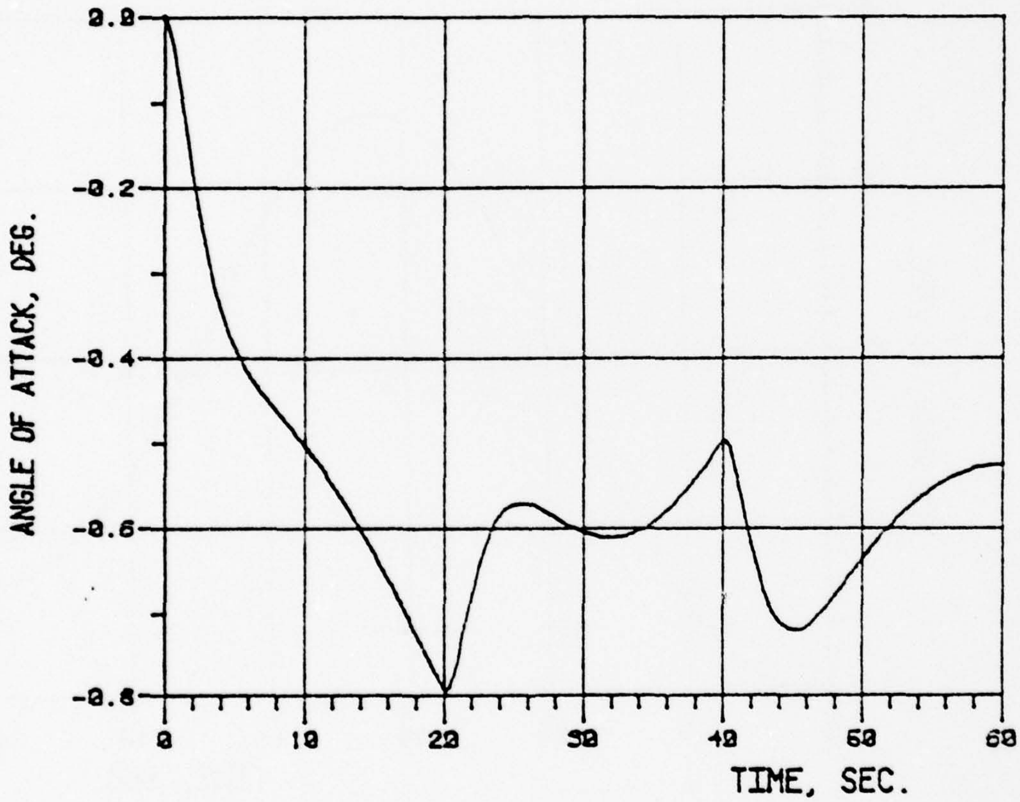


Figure 5.2c Angle of attack trajectory of the model: $x_M^3(t)$.

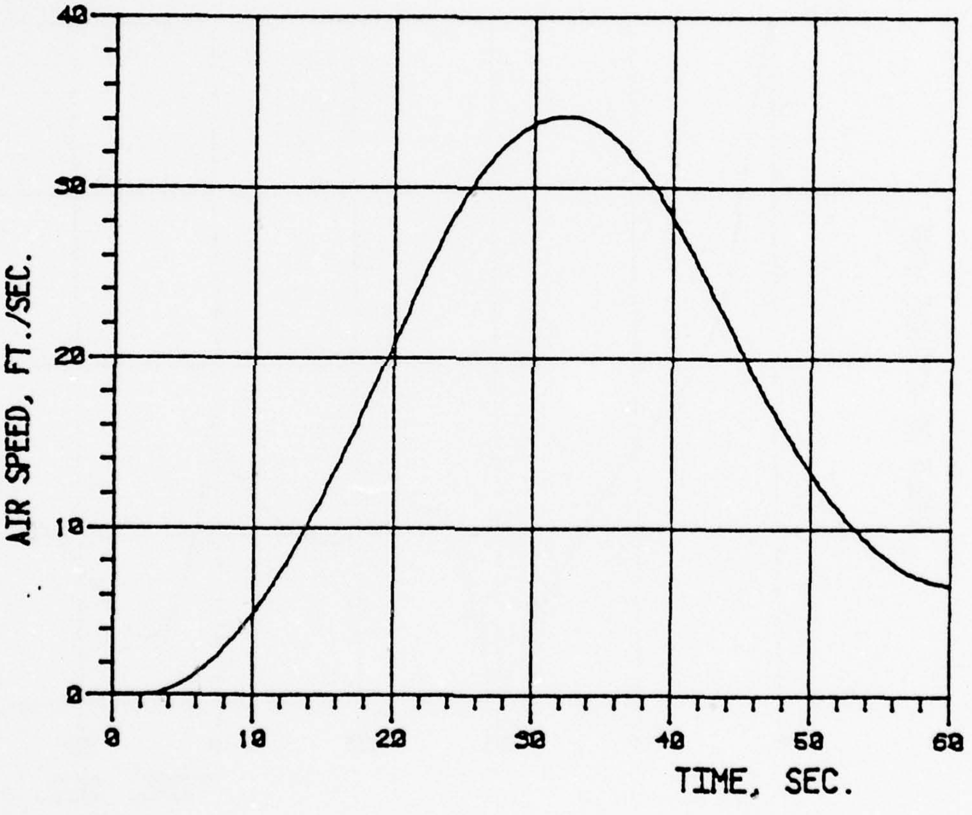


Figure 5.2d Air speed trajectory of the model: $x_M^4(t)$.

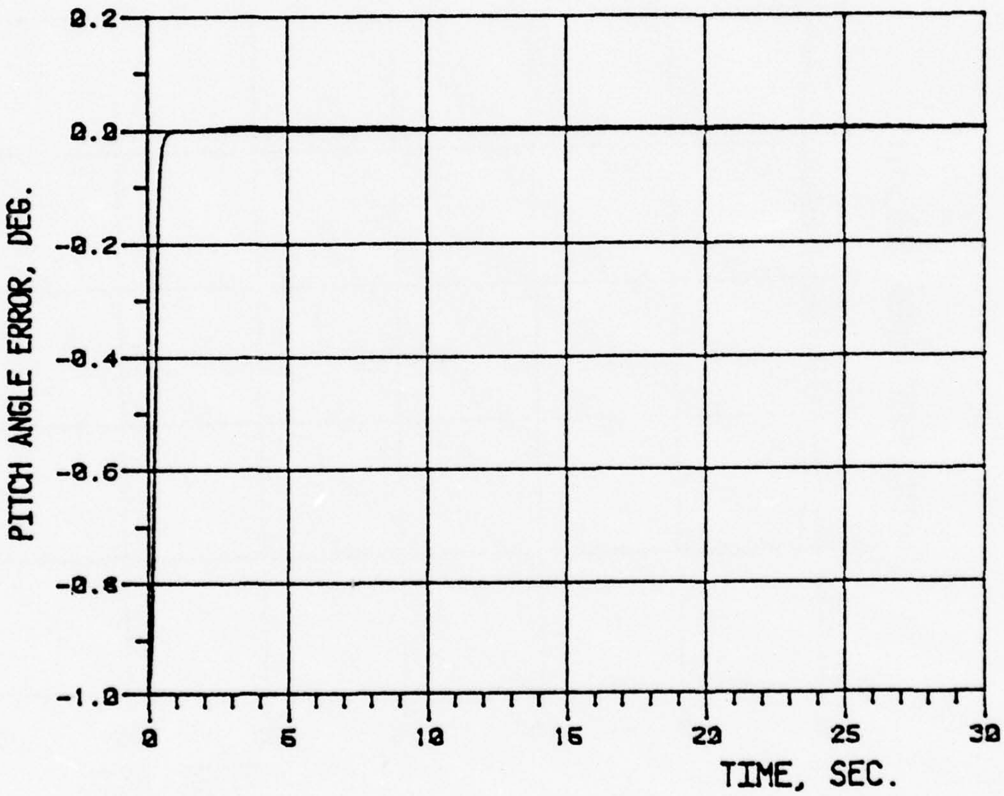


Figure 5.3a Plant model error response of pitch angle: $e_1(t)$.

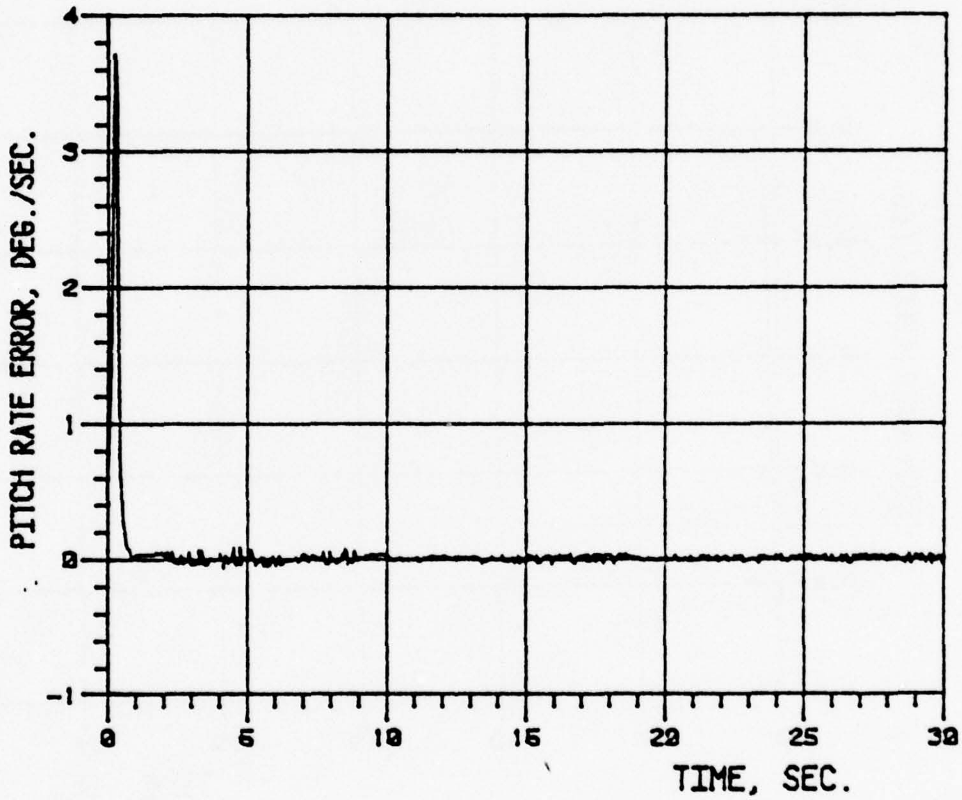


Figure 5.3b Plant model error response of pitch rate: $e_2(t)$.

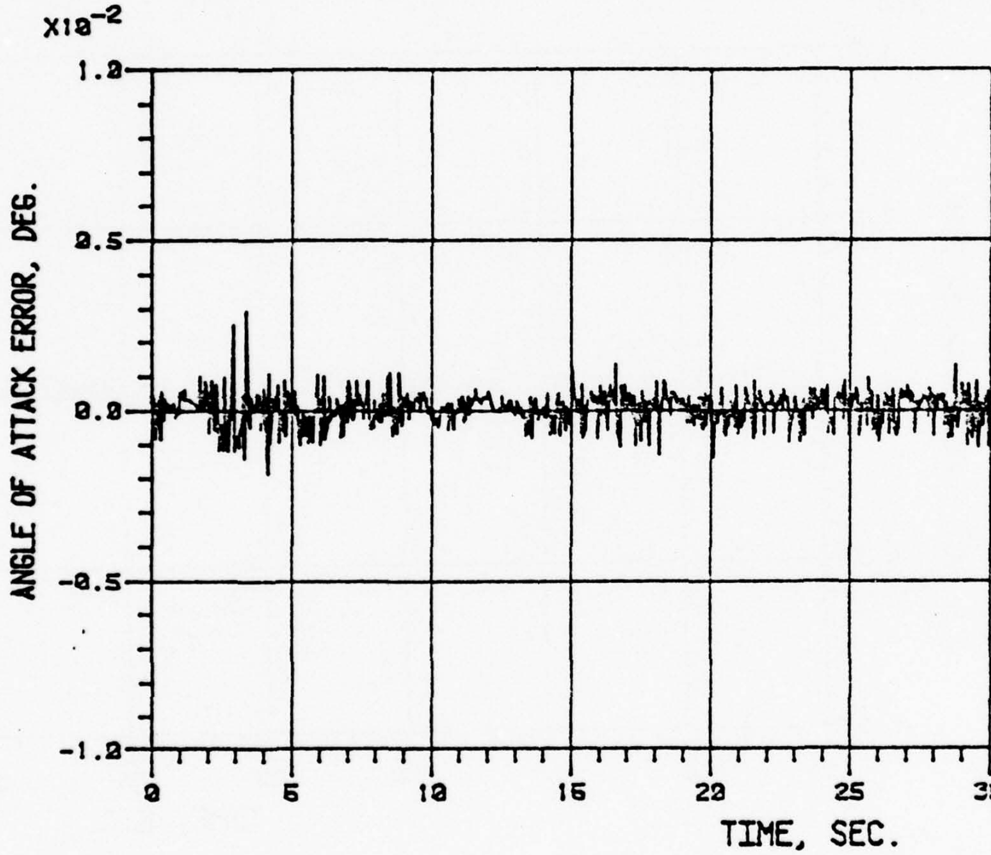


Figure 5.3c Plant model error response of angle of attack, $e_3(t)$.

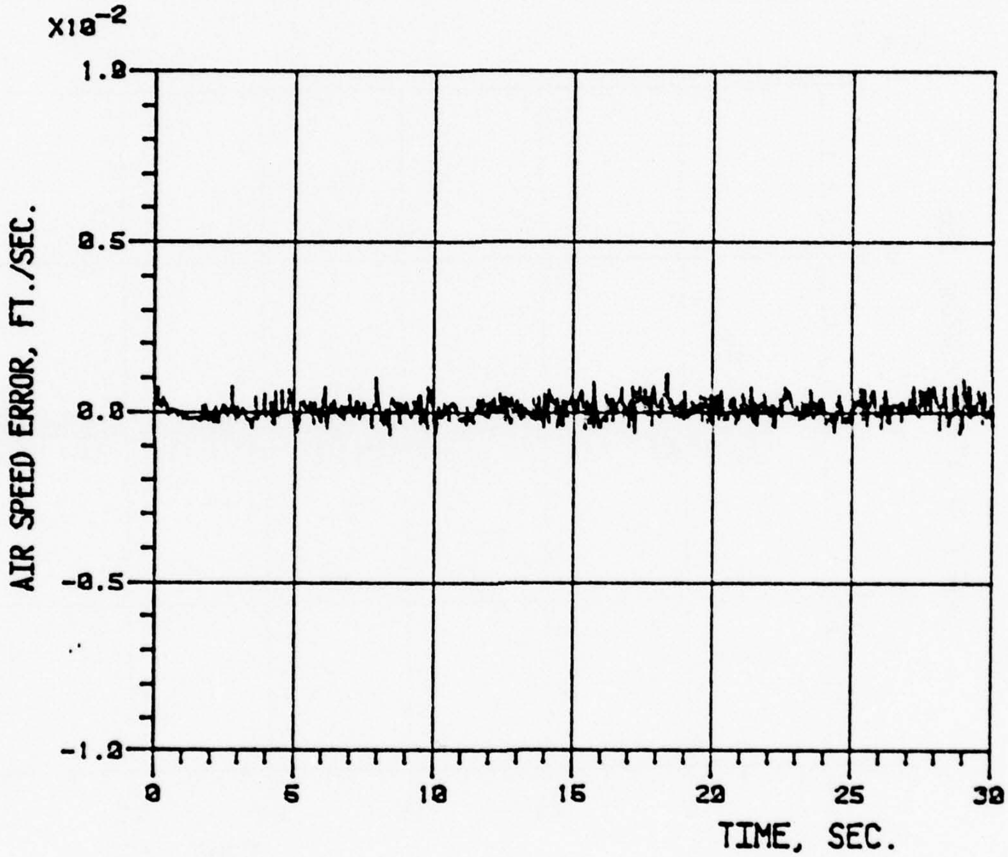


Figure 5.3d Plant model error response of air apeed: $e_4(t)$.

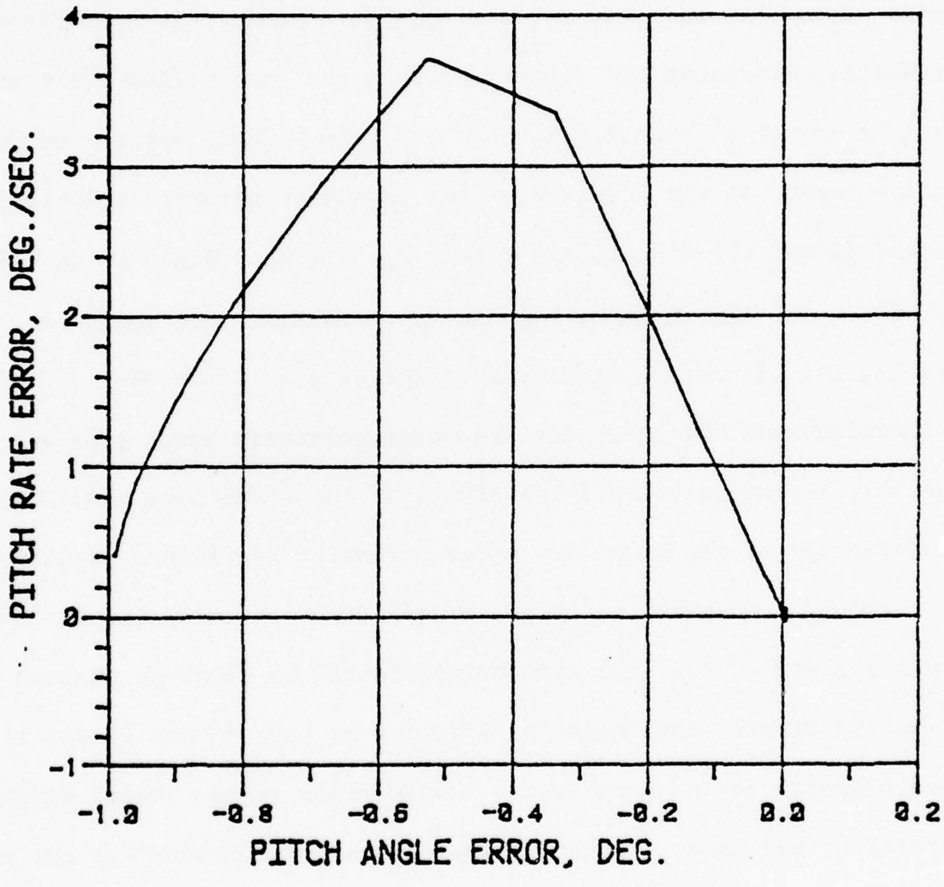


Figure 5.4 Phase trajectory in the (e_1,e_2) plane.

CHAPTER 6

CONCLUSIONS

In this work, we have developed to use singular perturbation methodology to analyze high gain feedback systems by decomposing the motion into slow and fast motions. We have shown that the effects of parameter variations and disturbances on the fast motion are attenuated by a factor of g^{-1} where g is the loop gain. Their effects on the slow motion depend on the structure of the parameter variations matrices δA and δB and the disturbance matrix E_o. If the ranges of δA, δB and E_o lie in the range of B_o, the input matrix, then these effects are also attenuated by a factor of g^{-1}. This new methodology has not only clarified such geometric properties but also made possible a semi-quantitative analysis of the robustness properties. The preservation of the stability under parameter variations is analyzed in terms of preservations of the stability of the slow and fast motions. The stability of the fast motion is affected by δB alone whereas the stability of the slow motion is affected by both δA and δB and its robustness property is the same as the transmission zeros. Using singular perturbation techniques, we have obtained the lower bounds for the gain factor that validate our results. These bounds indicate that the gain factor need not be extremely large. It need only to be large enough such that a separation of time scales takes place.

These analysis results are then used to develop new control structures. In the case when only some of the state variables are accessible, we have developed an observer with high gain feedback loops. It is the dual of a high gain state feedback system. The high gain

feedback matrix, in this case, is the measurement matrix C_o and the "input" matrix H is a design parameter. The class of parameter variations and disturbances which can be rejected are those that lie in the range of H. This observation leads to a design procedure for matrix H such that the range of H encompasses the ranges of δA, δB and E_o while maintaining the two time scale property and stability of the high gain observer. For the feedback system with two high gain loops: high gain feedback from a high gain observer, we have shown that the effects of parameter variations and disturbances on the fast motion are the same as when high gain state feedback is applied. The effects on the slow motion depends on the structures of δA, δB and E_o. If the ranges of δA, δB and E_o lie in the range of B_o and also in the range of H, then the effects are attenuated by a factor of g^{-1}. Furthermore, we have shown that the separation property of linear observer feedback systems does not always hold. This property holds only when the observer loop gain $\hat{g}$ is relatively larger than g and the slow motion is $O(g^{-1})$ sensitive to the parameter variations and disturbances. If only the sensitivity of the regulated variables are of concern, then the structural constraints on δA, δB and E_o are relaxed to a condition regarding the "controllability" of the slow motion with respect to the disturbance inputs. If the motion of the regulated variables contains only those slow modes which are not excited by the disturbances, then they are $O(g^{-1})$ sensitive to the disturbances. When the information concerning the ranges of δA, δB and E_o is difficult to ascertain, then by assuming the possibility of the worst case, this condition resolves to simply that the row space of the regulation matrix D_o lies in the row space of the high

gain feedback matrix K. This "worst case" condition is valid in both cases when the high gain state feedback and high gain feedback from high gain observers are applied. Moreover, it is the dual of the insensitivity condition for high gain observer, that is, the range of E_o lies in the range of H. Thus, taking into account the "worst case" structural constraint, the design procedure for matrix H developed earlier is applicable to the design of matrix K.

A comparative analysis of VSS and high gain feedback systems has been made. By showing that sliding mode in VSS is identical to the limiting slow motion of high gain state feedback systems as gain g tends to infinity, we have established that VSS and high gain state feedback systems reject the same class of parameter variations and disturbances. These geometric properties of high gain feedback systems yield new invariancy conditions for sliding mode. By drawing an analogy between VSS and high gain feedback systems, new control structures for VSS are developed. We have shown that Luenburger observers remain effective for variable structure feedback. In this case, the switching planes are defined in the observer state space and the motion of the resulting observer feedback system is close to the motion of VSS with switching planes defined in the system state space. In parallel to the development of high gain observers, we have designed observers with variable structures. In this case, the switching planes are defined in the observation error space and we have shown that variable structure observers reject the same class of disturbances and parameter variations as high gain

observers. Furthermore, the insensitivity property of VSS with variable structure observers is identical to the "two-high-gain-loops" systems. Finally, we establish that the sensitivity of the regulated variables in VSS is as in high gain feedback systems.

We have demonstrated that for the purpose of sensitivity reduction and disturbance rejection, variable structure feedback is an alternative structure to high gain feedback. Nevertheless a fundamentally important difference between these two classes of feedback systems is that their motions are significantly different outside the intersections of the switching planes or equivalently the null space of the high gain feedback matrix. In high gain feedback systems, this is the fast motion which is rapidly descending to this null space or escapes to infinity. In VSS, the characteristics of this motion depend on the design method used for the variable structure control. If the hierarchy of controls method is employed, then its trajectory descends to a one dimensional subspace of this null space (which is a switching plane), then to the intersections of two switching planes and finally to the intersectons of all the switching planes (the whole null space). Moreover, this motion is not necessarily fast. Our investigation on the robustness property of high gain feedback systems and VSS with respect to the neglection of actuator and sensor dynamics reveals that the capability to reach this null space is susceptible to the neglected small time constants if the reaching motion is forced to be fast by the use of large feedback gains. However, since high gain is not necessary in the design of variable structure feedback, reaching can be guaranteed in VSS. Hence, VSS is more robust with respect to sensor and actuator dynamics.

In conclusion, we emphasize that, in this study, we have not considered the nature of the disturbance inputs. If we know the type of disturbance inputs, then compensator feedback can be incorporated together with high gain or variable structure feedback. For example, in the case of constant disturbances, we can apply high gain feedback to reduce the steady state errors as much as possible. If, owing to the system structure, high gain feedback cannot make the system motion totally insensitive to disturbances, the next step is to incorporate integral feedback to remove the remaining steady state errors. This amounts to a combination of fast and slow control actions where the sole purpose of the slow control action (integral feedback) is to improve the low frequency response and the steady state characteristics. We comment that combinations of high gain or variable structure feedback and compensator feedback would significantly improve the transient responses and we foresee it to be a fruitful approach in control system designs.

REFERENCES

1. M. V. Meerov, "Automatic Control Systems Stable with Indefinitely Large Values of Gain," Automatika i Telemekhanika, Vol. 8, No. 4, 1947.

2. W. R. Evans, "Graphical Analysis of Control Systems," Trans. AIEE, Vol. 67, 1948, pp. 547-551.

3. H. W. Bode, Network Analysis and Feedback Amplifier Design, Van Nostrand, Princeton, N.J., 1945.

4. I. M. Horowitz, Synthesis of Feedback Systems, Academic Press, New York, 1963.

5. E. J. Davison and S. H. Wang, "Properties and Calculations of Transmission Zeros of Linear Multivariable Systems," Automatica, Vol. 10, No. 6, 1974, pp. 643-658.

6. B. Kouvaritakis and A. G. J. MacFarlane, "Geometric Approach to Analysis and Synthesis of System Zeros (Part 1): Square Systems," International Journal of Control, Vol. 23, No. 2, 1976, pp. 149-166.

7. U. Shaked and B. Kouvaritakis, "Asymptotic Behavior of Root-Locus of Linear Multivariable Systems," International Journal of Control, Vol. 23, No. 3, 1976, pp. 297-340.

8. J. L. Willems, "Disturbance Isolation in Linear Feedback Ssytems," International Journal System Science, Vol. 6, No. 3, 1975, pp. 233-238.

9. U. Shaked and B. Kouvaritakis, "The Zeros of Linear Optimal Control Systems and their Role in High Gain Stability Design," IEEE Trans. on Automatic Control, Vol. AC-22, August 1977, pp. 597-599.

10. E. J. Davison, "The Decentralized Stabilization and Control of a Class of Unknown Nonlinear Time-Varying Systems," Automatica, Vol. 10, No. 3, 1974, pp. 309-316.

11. A. Jameson and R. E. O'Malley, Jr., "Cheap Control of the Time-Invariant Regulator," Applied Mathematics and Optimization, Vol. 1, No. 4, 1975, pp. 337-354.

12. H. Kwakernaak and R. Sivan, "The Maximally Achievable Accuracy of Linear Optimal Regulators and Linear Optimal Filters," IEEE Trans. on Automatic Control, Vol. AC-17, No. 1, February 1972, pp. 79-86.

13. H. Kwakernaak, "Asymptotic Root Loci of Multivariable Linear Optimal Regulators," IEEE Trans. on Automatic Control, Vol. AC-21, No. 3, June 1976, pp. 378-382.

14. S. V. Emelyonov, and I. A. Burovoi, et al., Mathematical Models of Industrial Processes and Design of Variable Structure Control Systems, Metalurgia, Moscow, 1964 (in Russian).

15. S. V. Emelyanov, Variable Structure Control Systems, Nauka, Moscow, 1967 (in Russian); also Oldenburg Verlag Munchen-Wien 1969 (in German).

16. S. V. Emelyanov, Variable Structure Systems and their Applications to Flight Automation, Nauka, Moscow, 1968 (in Russian).

17. S. V. Emelyanov, Theory of Variable Structure Systems, Nauka, Moscow, 1970 (in Russian).

18. U. Itkis, Control Systems of Variable Structure, Wiley, New York 1976.

19. V. I. Utkin, "Variable Structure Systems with Sliding Mode: A Survey," IEEE Trans. on Automatic Control, Vol. AC-22, April 1977, pp. 212-222.

20. D. D. Siljak and S. K. Sundareshan, "Large-Scale Systems: Optimality Vs. Reliability," Directions in Decentralized Control, Many-person Optimization and Large-Scale Systems, Plenum, New York, 1976, pp. 411-425.

21. P. V. Kokotovic and P. Sannuti, "Singular Perturbation Method for Reducing the Model Order in Optimal Control Design," IEEE Trans. on Automatic Control, Vol. AC-13, No. 4, August 1968, pp. 377-384.

22. Singular Perturbations: Order Reduction in Control System Design, American Society of Mechanical Engineers, New York, N.Y. 1972.

23. P. V. Kokotovic, R. E. O'Malley, Jr., and P. Sannuti, "Singular Perturbations and Order Reduction in Control Theory-An Overview," Automatica, Vol. 12, pp. 123-132, March 1976.

24. P. V. Kokotovic, "A Riccati Equation for Block-Diagonalization of Ill-Conditioned System," IEEE Trans. on Automatic Control, Vol. AC-20, No. 6, December 1975, pp. 812-814.

25. P. V. Kokotovic and A. H. Haddad, "Controllability and Time-Optimal Control of Systems with Slow and Fast Modes," IEEE Trans. on Automatic Control, Vol. AC-20, No. 1, February 1975, pp. 111-113.

26. R. Sivan, "On Zeroing the Output and Maintaining it Zero," IEEE Trans. on Automatic Control, Vol. AC-10, No. 2, pp. 193-194.

27. K. K. D. Young, "Multiple Time Scales in Single Input-Single Output High Gain Feedback Systems," to appear in Proceeding of the 15th Annual Allerton Conference on Communication, Control and Computing, Sept. 28-30, 1977.

28. G. S. Axelby and E. J. Davison, "On the Computation of Transmission Zeros of Linear Multivariable Systems," Automatica, Vol. 12, No. 5, 1976, pp. 533-534.

29. U. Shaked, "Design Techniques for High Feedback Gain Stability," International Journal of Control, Vol. 24, No. 1, 1976, pp. 137-144.

30. B. Kouvanitakis and A. G. J. MacFarlane, "Geometric Approach to Analysis and Synthesis of System Zeros Part 2: Non-Square Systems," International Journal of Control, Vol. 23, No. 2, pp. 167-181.

31. H. H. Rosenbrock, State-Space and Multivariable Theory, Wiley, New York 1970.

32. R. E. O'Malley, Jr. and A. Jameson, "Singular Perturbations and Singular Arcs-Part I, II," IEEE Trans. on Automatic Control, Vol. AC-20, No. 2, April 1975, pp. 218-226; Vol. AC-22, No. 3, June 1977, pp. 328-337.

33. W. M. Wonham, Linear Multivariable Control: A Geometric Approach (lecture notes in economics and mathematical systems), Vol. 101, Springer, Berlin, 1974.

34. B. D. O. Anderson and J. B. Moore, Linear Optimal Control, Prentice-Hall, Englewood Cliffs, N.J., 1971.

35. J. H. Chow and P. V. Kokotovic, "A Decomposition of Near-Optimum Regulators for Systems with Slow and Fast Modes," IEEE Trans. on Automatic Control, Vol. AC-21, No. 5, October 1976, pp. 701-705.

36. K. Martensson, "On the Matrix Riccati Equation," Information Science, Vol. 3, 1971, pp. 17-49.

37. V. I. Utkin, Sliding Modes and Their Application to Variable Structure Systems, Nauka, Moscow, 1974 (in Russian), to be published in English by Mir., Moscow, 1977.

38. V. I. Utkin, "Equations of Sliding Mode in Discontinuous Systems, I,II," Automation and Remote Control, No. 12, 1971, pp. 1897-1907; No. 2, 1972, pp. 211-219.

39. B. Drazenovic, "The Invariance Conditions in Variable Structure Systems," Automatica, Vol. 5, pp. 287-295, 1969.

40. Ya. Z. Tsyphin, The Theory of Automatic Control Relay Systems, Gostekhizhat, 1955 (in Russian).

41. M. V. Meerov, Structural Synthesis of High-Accuracy Automatic Control Systems, Pergamon Press, Oxford 1965.

42. S. V. Emelyanov and V. A. Taran, "On the Use of inertial sections in designing a certain class of automatic control system with a variable structure, I, II," Automation and Remote Control, No. 1, pp. 29-42; No. 2, pp. 183-190, 1963.

43. V. A. Taran, "Control of a linear object by a variable-structure Astatic Control Without the Use of Pure Derivatives in Control Law, I, II," Automation and Remote Control, No. 10, pp. 1278-1288; No. 11, pp. 1406-1412, 1964.

44. V. B. Ezerov and V. I. Utkin, "A variable structure filter as a phase-lead network," Variable Structure Systems and their Application to Flight Automation, Nauka, Moscow 1968, pp. 171-182 (in Russian.

45. S. V. Emelyanov and V. A. Taran, "Use of passive switched filters in variable structure control systems," Engineering Cybernetics, No. 5, pp. 143-149, 1963.

46. S. V. Emelyanov and V. A. Taran, "Stabilization of variable structure control systems by means of inertial elements with variable time constants " Automation and Remote Control, No. 6, pp. 790-784, 1964.

47. V. A. Taran, "Construction of Control Systems Using Switched Phase Shift Filters," Engineering Cybernetics, No. 4, pp. 172-180, 1965.

48. H. Kwakernaak and R. Sivan, Linear Optimal Control Systems, Wiley-Interscience, New York 1972.

49. S. H. Wang and E. J. Davison, "On the stabilization of Decentralized Control Systems," IEEE Trans. on Automatic Control, Vol. AC-18, no. 5, October 1973, pp. 473-478.

50. Ü. Özgüner, "The Analysis and Control of Large Scale Composite Systems," Report R-680, Coordinated Science Laboratory, University of Illinois, Urbana, May 1975.

51. F. G. Shinskey, Process-Control Systems, McGraw-Hill, 1967, Chapter 11.

52. W. L. Luyben, Process Modeling, Simulation, and Control for Chemical Engineers, McGraw-Hill, 1973.

53. E. J. Davison, "Control of a Distillation Column with Pressure Variation," Trans. Instn. Chem. Engrs. Vol. 45, 1967, pp. T229-T250.

54. H. H. Rosenbrock, "Distinctive Problems of Process Control," Chemical Engineering Progress, Vol. 58, no. 9, September 1962, pp. 43-50.

55. E. J. Davison and N. S. Rau, "The Optimal Output Feedback Control of a Synchronous Machine," IEEE Trans. on Power Apparatus and Systems, Vol. PAS-90, no. 5, September/October 1971, pp. 2123-2134.

56. C. A. Winsor and R. J. Roy, "The Application of Specific Optimal Control to Design of Desensitized Model Following Control Systems," IEEE Trans. on Auto. Control, Vol. AC-15, pp. 326-333, 1970.

57. I. D. Landau and B. Courtiol, "Adaptive Model Following for Flight Control and Simulation," J. Aircraft, Vol. 9, pp. 668-674, 1972.

58. I. D. Landau and B. Courtiol, "Design of Multivariable Adaptive Model Following Control Systems," Automatica, Vol. 10, pp. 483-494, 1974.

59. System/360, Scientific Subroutine Package, Version III, Programmer's Manual, IBM Publication 360A-CM-03X, 1970.

APPENDIX A

TRANSFORMATION OF HIGH GAIN FEEDBACK TO SINGULARLY PERTURBED SYSTEM

Consider the transformation $\tilde{x} = Tx$ as two successive transformations, that is

$$x' = Mx, \quad \tilde{x} = \Gamma x'. \tag{A1}$$

Denote

$$MA_oM^{-1} = \begin{bmatrix} A^o_{11} & A^o_{12} \\ A^o_{21} & A^o_{22} \end{bmatrix} \tag{A2}$$

then the system

$$\dot{x} = MA_oM^{-1}x + MB_ou \tag{A3}$$

by (A2) and (2.9) is (2.51), (2.52). From (2.17),

$$\Gamma^{-1} = \begin{bmatrix} I_{n-m} & 0 \\ -C_2^{-1}C_1 & C_2^{-1} \end{bmatrix} \tag{A4}$$

with C_1, C_2 as in (2.12). The system

$$\dot{\tilde{x}} = \Gamma MA_oM^{-1}\Gamma^{-1}\tilde{x} + \Gamma MB_ou \tag{A5}$$

by (A2) and (A4) and $\tilde{x} = [\tilde{x}_1 \quad \tilde{x}_2]^T$ is (2.22), (2.23) with

$$F_{11} = A^o_{11} - A^o_{12}C_2^{-1}C_1 \tag{A6}$$

$$F_{12} = A^o_{12}C_2^{-1} \tag{A7}$$

$$H_1 = C_1A^o_{11} + C_2A^o_{21} - H_2C_1 \tag{A8}$$

$$H_2 = (C_1A^o_{12} + C_2A^o_{22})C_2^{-1}. \tag{A9}$$

Denote

$$M = \begin{bmatrix} M_1 \\ \\ M_2 \end{bmatrix}, \quad M^{-1} = \quad [S_1 \quad S_2] \tag{A10}$$

where M_1, S_1^T are $(n-m) \times n$ and M_2, S_2^T are $m \times n$ matrices, then

$$S_1M_1 + S_2M_2 = I_n. \tag{A11}$$

From (2.12),

$$C_1 = C_oS_1, \; C_2 = C_oS_2 \tag{A12}$$

$$B_2^o = M_2B_o, \; 0 = M_1B_o. \tag{A13}$$

In terms of (A10), A^o_{11} in (A2) becomes

$$A^o_{11} = M_1A_oS_1. \tag{A14}$$

Substituting into (A6) yields (3.38). Using the expressions (A11)-(A13), $x = M^{-1}\Gamma^{-1}\tilde{x}$ is

$$x_o = (S_1 - S_2C_2^{-1}C_1)\tilde{x}_1 + B_o(C_oB_o)^{-1}\tilde{x}_2. \tag{A15}$$

From (A12)

$$C_o(S_1 - S_2C_2^{-1}C_1) = 0 \tag{A16}$$

and by letting

$$N = S_1 - S_2C_2^{-1}C_1 \tag{A17}$$

(A15) is (2.18). By (2.12) and (A10), $\tilde{x} = \Gamma Mx$ becomes

$$\begin{bmatrix} \tilde{x}_1 \\ \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} M_1x \\ \\ (C_1M_1 + C_2M_2)x \end{bmatrix} = \begin{bmatrix} M_1x \\ \\ C_ox \end{bmatrix} \tag{A18}$$

APPENDIX B

PROOF OF THEOREM 2.1

For the system (2.22), (2.33), by Lemma 1 in [24], if assumption (2.14) is satisfied and if μ satisfies

$$\mu a \leq \frac{1}{3}(b + 2\mu acd) \tag{B1}$$

where $a = \| (C_oB_o)^{-1} \|$, $b = \| F_{11} \|$, $c = \| F_{12} \|$ and $d = \| H_1 \|$, then L of (2.25) exists. $L = L_o + \mu G$ where

$$L_o = (C_oB_o + \mu H_2)^{-1}H_1. \tag{B2}$$

From (B1), μ_c in (2.27) is obtained,

$$\mu_c = \frac{(b^2 + \frac{8}{3}cd)^{\frac{1}{2}} - b}{4acd} . \tag{B3}$$

By application of the implicit function theorem to

$$\begin{aligned}(C_oB_o + \mu H_2) - \mu GF_{11} + \mu^2 GF_{12}L_o + \mu^2 L_oF_{12}G \\ + \mu^3 GF_{12}G = L_oF_{11} - \mu L_oF_{12}L_o\end{aligned} \tag{B4}$$

we can show that G possesses a power series at $\mu = 0$, that is,

$$G = \sum_{i=0}^{k} G_i\mu^i + O(\mu^k). \tag{B5}$$

A recursive formula for calculating G_i is obtained by substituting (B5) into (B4) and equating coefficients of powers of μ,

$$G_o = (C_oB_o + \mu H_2)^{-1}L_oH_1F_{11} \tag{B6}$$

$$G_1 = (C_oB_o + \mu H_2)^{-1}[G_oF_{11} - L_oF_{12}L_o] \tag{B7}$$

$$G_k = (C_oB_o + \mu H_2)^{-1}[G_{k-1}F_{11} - G_{k-2}F_{12}L_o - L_oF_{12}G_{k-2} - \sum_{j=0}^{k-3} G_jF_{12}G_{k-j-3}] \quad \text{for } k \geq 2 \text{ with } \sum_{j=0}^{-1} \equiv 0. \tag{B8}$$

APPENDIX C

APPROXIMATE SOLUTIONS OF SINGULARLY PERTURBED SYSTEMS

In the following lemmas, it is assumed that the matrix Σ_{22} is a stable matrix and there exist positive constants c_1 and c_2 such that $|\omega(t)| \leq c_1$ and $|\dot{\omega}(t)| \leq c_2$.

<u>Lemma C1</u>:

The solution of the singularly perturbed system

$$\dot{\xi} = \Sigma_{11}\xi + \Sigma_{12}\sigma + \Pi_1\omega \tag{C1}$$

$$\mu\dot{\sigma} = \Sigma_{22}\sigma + \upsilon\Pi_2\omega \tag{C2}$$

can be approximated by

$$\xi(t) = e^{\xi_{11}t}\xi(0) + \int_0^t e^{\Sigma_{11}(t-\tau)}\Pi_1\omega(\tau)dt + 0(\mu) \tag{C3}$$

$$\sigma(t) = e^{\Sigma_{22}\frac{t}{\mu}}\sigma(0) + 0(\mu). \tag{C4}$$

<u>Proof</u>: By application of the variation of parameter formula to (C2), we have

$$\sigma(t) = e^{\Sigma_{22}\frac{t}{\mu}}\sigma(0) + \int_0^t e^{\Sigma_{22}\frac{(t-\mu)}{\mu}}\Pi_1\omega(\tau)d\tau. \tag{C5}$$

Integrating the integral by parts, (C5) becomes

$$\sigma(t) = e^{\Sigma_{22}\frac{\tau}{\mu}}\sigma(0) + \mu[\Sigma_{22}^{-1}\, e^{\Sigma_{22}\frac{t}{\mu}}\Pi_1\omega(0) - \Sigma_{22}^{-1}\Pi_1\omega(t)$$

$$+ \int_0^t \Sigma_{22}^{-1}\, e^{\Sigma_{22}\frac{(t-\tau)}{\mu}}\,\Pi\,\dot{\omega}(\tau)d\tau] \qquad \text{(C6)}$$

Since by assumption $|\omega(t)| \leq c_1$ and $|\dot{\omega}(t)| \leq c_2$, (C4) is (C6).

Applying the variation of parameters formula to (C1), we obtain

$$\xi(t) = e^{\Sigma_{11}t}\xi(0) + \int_0^t e^{\Sigma_{11}(t-\tau)}\Sigma_{12}e^{\Sigma\frac{\tau}{\mu}}\sigma(0)d\tau$$

$$+ \int_0^t e^{\Sigma_{11}(t-\tau)}\Pi_1\omega(\tau)d\tau. \qquad \text{(C7)}$$

Integrating the second integral in (C7) by parts, we have

$$\xi(t) = e^{\Sigma_{11}t}\xi(0) + \mu[\Sigma_{12}\Sigma_{12}^{-1}\sigma(t) - e^{\Sigma_{11}t}\Sigma_{12}\Sigma_{22}^{-1}\sigma(0)$$

$$+ \int_0^t \Sigma_{11} e^{\Sigma_{11}(t-\tau)}\Sigma_{12}\Sigma_{22}^{-1}\sigma(\tau)d\tau + \int_0^t e^{\Sigma_{11}(t-\tau)}\Pi_1\omega(\tau)d\tau. \qquad \text{(C8)}$$

This concludes the proof.

<u>Lemma C2</u>:

The solution of the singularly perturbed system

$$\dot{\xi} = \Sigma_{11}\xi + \Sigma_{12}\sigma^{11} + \Pi_1\omega \qquad \text{(C9)}$$

$$\mu\dot{\sigma}'' = \Sigma_{21}\xi + \Sigma_{22}\sigma'' + \mu\Pi_2\omega \qquad \text{(C10)}$$

can be approximated by

$$\xi(t) = e^{(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})t}\xi(0) + \int_0^t e^{(\Sigma_{11} - \Sigma_{12}\Sigma_{22}\Sigma_{21})(t-\tau)}\Pi_1\omega(\tau)d\tau + O(\mu) \tag{C11}$$

$$\sigma''(t) = e^{\Sigma_{22}\frac{t}{\mu}}\sigma''(0) + e^{\Sigma_{22}\frac{t}{\mu}}\Sigma_{22}^{-1}\Sigma_{21}\xi(0) - \Sigma_{22}^{-1}\Sigma_{21}\xi(t) + O(\mu) \tag{C12}$$

Proof: Let

$$\sigma' = \Sigma_{22}^{-1}\Sigma_{21}\xi + \sigma'' \tag{C13}$$

then (C7),(C8) becomes

$$\dot{\xi} = (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})\xi + \Sigma_{12}\sigma' + \Pi_1\omega \tag{C14}$$

$$\begin{aligned}\mu\dot{\sigma}' = \mu\Sigma_{22}^{-1}\Sigma_{21}(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})\xi + (\mu\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{12} + \Sigma_{22})\sigma' \\ + \mu(\Sigma_{22}^{-1}\Sigma_{21}\Pi_1 + \Pi_2)\omega.\end{aligned} \tag{C15}$$

Introduce the variable

$$\sigma = \sigma' + \mu\Gamma\xi \tag{C16}$$

where

$$\Gamma = (\Sigma_{22} + \mu\Sigma_{21}\Sigma_{12}\Sigma_{22}^{-1})^{-1}\Sigma_{21}(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}) + \mu\Omega \tag{C17}$$

and matrix Ω possesses a power series at $\mu = 0$. Its coefficients can be calculated from a recursive formula similar to (B6)-(B8). By [6,7], we obtain a equivalent block triangular system

$$\dot{\xi} = [\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} - \mu\Sigma_{12}\Gamma]\xi + \Sigma_{12}\sigma + \Pi_1\omega \tag{C18}$$

$$\mu\dot{\sigma} = [\Sigma_{22} + \mu\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{12} + \mu^2\Gamma\Sigma_{12}]\sigma + \mu(\Pi_2 + \Sigma_{22}^{-1}\Sigma_{21}\Pi_1)\omega. \tag{C19}$$

This system is now in the same form as (C1), (C2). Hence (C11) and

$$\sigma(t) = e^{\Sigma\frac{t}{\mu}}\sigma(0) + 0(\mu) \tag{C20}$$

are obtained by Lemma C1. From (C13) and (C16).

$$\sigma''(t) = \sigma(t) - \Sigma_{22}^{-1}\Sigma_{21}\xi(t) - \mu\Gamma\xi(t). \tag{C21}$$

Substituting (C21) into (C20), we obtain (C12). This concludes the proof.

<u>Lemma C3</u>:

The solution of the singularly perturbed system

$$\dot{\xi}' = \Sigma_{11}\xi' + \frac{\Sigma_{12}}{\mu}\psi' + \Pi_1\omega \tag{C22}$$

$$\dot{\psi}' = \Sigma_{21}\xi' + \frac{\Sigma_{22}}{\mu}\psi' + \Pi_2\omega \tag{C23}$$

can be approximated by

$$\begin{aligned}\xi'(t) = {} & e^{(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})t}\xi'(0) - e^{(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})t}\Sigma_{12}\Sigma_{22}^{-1}\psi'(0) \\ & + \Sigma_{12}\Sigma_{22}^{-1}\psi'(t) + \int_0^t e^{(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})(t-\tau)}[\Pi_1-\Sigma_{12}\Sigma_{22}^{-1}\Pi_2]\omega(\tau)d\tau \\ & + 0(\mu)\end{aligned} \tag{C24}$$

$$\psi'(t) = e^{\Sigma_{22}\frac{t}{\mu}}\psi'(0) + O(\mu). \tag{C25}$$

Proof: Let

$$\xi = \xi' - \Sigma_{12}\Sigma_{22}^{-1}\psi' \tag{C26}$$

then (C22),(C23) becomes

$$\dot{\xi} = (\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})\xi + (\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})\Sigma_{12}\Sigma_{21}\psi' + (\Pi_1-\Sigma_{12}\Sigma_{22}^{-1}\Pi_2)\omega \tag{C27}$$

$$\mu\dot{\psi}' = \mu\Sigma_{21}\xi' + (\mu\Sigma_{21}\Sigma_{12}\Sigma_{22}^{-1}+\Sigma_{22})\psi' + \mu\Pi_2\omega \tag{C28}$$

which is the same form as (C14), (C15). Hence by Lemma C2, (C25) and

$$\xi(t) = e^{(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})t}\xi(0) + \int_0^t e^{(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})(t-\tau)} \cdot$$

$$[\Pi_1-\Sigma_{12}\Sigma_{22}^{-1}\Pi_2]\omega(\tau)d\tau + O(\mu) \tag{C29}$$

are obtained. From (C26)

$$\xi'(t) = \xi(t) + \Sigma_{12}\Sigma_{22}^{-1}\psi'(t). \tag{C30}$$

Substituting (C30) into (C29), we obtain (C24). This completes the proof.

APPENDIX D

TRANSFORMATION FOR A "TWO-HIGH-GAIN-LOOPS" SYSTEMS

In order to investigate the behavior of the high gain observer feedback system (3.57), (3.58), it is convenient to transform it into singularly perturbed systems of Appendix C. Let $\mathcal{M}$ be an $2n \times 2n$ matrix,

$$\mathcal{M} = \begin{bmatrix} M_\delta & 0_{n\times n} \\ \hat{M}_\delta & \hat{M} \end{bmatrix} \tag{D1}$$

where $\hat{M}$ is as defined in (3.38), M_δ and $\hat{M}_\delta$ are $n\times n$ matrices such that

$$M_\delta(B_o + \delta B) = \begin{bmatrix} 0_{(n-m)\times m} \\ B_2 \end{bmatrix} \tag{D2}$$

$$\hat{M}_\delta(B_o + \delta B) + \hat{M}\delta B = 0. \tag{D3}$$

By the assumption that rank $(B_o + \delta B) = m$ for all δB, M_δ exists and there exists $\hat{M}_\delta$ that satisfies (D3)

$$\hat{M}_\delta = -\hat{M}S \tag{D4}$$

where S is a solution of the matrix equation,

$$SB = \delta B. \tag{D5}$$

We define

$$B = B_o + \delta B \tag{D6}$$

for convenience. Introducing the transformation

$$\begin{bmatrix} x_{11}'' \\ x_2'' \\ \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = M \begin{bmatrix} x \\ e' \end{bmatrix} \tag{D7}$$

with $x_1'' \in R^{n-m}$, $x_2'' \in R^m$, $\varepsilon_1 \in R^{n-r}$ and $\varepsilon_2 \in R^r$. In these new coordinates, system (3.57), (3.58) becomes

$$\dot{x}_1'' = A_{11}x_1'' + A_{12}x_2'' + E_1 f \tag{D8}$$

$$\dot{x}_2'' = A_{21}x_1'' + A_{22}x_2'' + g\, B_2[(K_1 + \tilde{K}_1)x_1'' + (K_2 + \tilde{K}_2)x_2''$$

$$-\hat{K}_1\varepsilon_1 - \hat{K}_2\varepsilon_2] + E_2 f \tag{D9}$$

$$\dot{\varepsilon}_1 = F_{11}x_1'' + F_{12}x_2'' + \hat{A}_{11}^o\varepsilon_1 + \hat{A}_{12}^o\varepsilon_2 + \hat{E}_1 f \tag{D10}$$

$$\dot{\varepsilon}_2 = F_{21}x_1'' + F_{22}x_2'' + \hat{A}_{21}^o\varepsilon_1 + \hat{A}_{22}\varepsilon_2 + \hat{g}H_2[-(\tilde{C}_1 - \delta C_1)x_1'' - (\tilde{C}_2 - \delta C_2)x_2''$$

$$+ C_1^o\varepsilon_1 + C_2^o\varepsilon_2] + \hat{E}_2 f \tag{D11}$$

where

$$M_\delta(A_o + \delta A)M_\delta^{-1} \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{D12}$$

$$\hat{M}_\delta(A_o + \delta A)M_\delta^{-1} + M_\delta A M_\delta^{-1} - \hat{M}A_o\hat{M}^{-1}\hat{M}_\delta M_\delta^{-1} \equiv \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \tag{D13}$$

AD-A057 462 ILLINOIS UNIV AT URBANA-CHAMPAIGN DECISION AND CONTROL LAB F/G 9/3
ANALYSIS AND SYNTHESIS OF HIGH GAIN AND VARIABLE STRUCTURE FEED--ETC(U)
NOV 77 K D YOUNG DAAB07-72-C-0259
UNCLASSIFIED DC-7 UILU-ENG-77-2247 NL

3 OF 3
AD
A057 462

END
DATE
FILMED
9-78
DDC

$$M_\delta E_o = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \quad (\hat{M}_\delta + \hat{M})E_o = \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \end{bmatrix} \tag{D14}$$

$$KM_\delta^{-1} = [K_1 \quad K_2] \qquad K\hat{M}^{-1}M_\delta\tilde{M}_\delta^{-1} = [\tilde{K}_1 \qquad \tilde{K}_2] \tag{D15}$$

$$KM^{-1} = [\hat{K}_2 \quad \hat{K}_2] \qquad C_o\hat{M}^{-1}\hat{M}_\delta M_\delta^{-1} = [\tilde{C}_1 \qquad \tilde{C}_2] \tag{D16}$$

$$\delta CM_\delta^{-1} = [\delta C_1 \quad \delta C_2]. \tag{D17}$$

and $\hat{A}^o_{ij}$, C^o_i the H_2 are as defined in (3.38)-(3.41).

We now introduce another transformation that decomposes the system (D8)-(D11) into slow and fast subsystems. Let

$$T = \begin{bmatrix} I_{2n-m-r} & 0 \\ K_1 & K_2 \end{bmatrix} \begin{bmatrix} I_{n-m} & 0 & 0 & 0 \\ 0 & 0 & I_{n-r} & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & I_r \end{bmatrix} \tag{D18}$$

where

$$K_1 = \begin{bmatrix} K_1 + \tilde{K}_1 & -\hat{K}_1 \\ -\tilde{C}_1 + \delta C_1 & C^o_1 \end{bmatrix}, \quad K_2 = \begin{bmatrix} K_2 + \tilde{K}_2 & -\hat{K}_2 \\ -\tilde{C}_2 + \delta C_2 & C^o_2 \end{bmatrix} \tag{D19}$$

In the new coordinates $(x_1', \varepsilon_1, \hat{\phi}, \hat{\eta})$,

$$\begin{bmatrix} x_1'' \\ \varepsilon_1 \\ \hat{\phi} \\ \hat{\eta} \end{bmatrix} = T \begin{bmatrix} x_1'' \\ x_2'' \\ \varepsilon_2 \\ \varepsilon_2 \end{bmatrix} \tag{D20}$$

system (8)-(11) is represented by

$$\begin{bmatrix} \dot{x}_1'' \\ \dot{\epsilon}_1 \end{bmatrix} = \left(A_1 - A_2K_2^{-1}K_1\right)\begin{bmatrix} x_1'' \\ \epsilon_1 \end{bmatrix} + A_2K_2^{-1}\begin{bmatrix} \hat{\phi} \\ \hat{\eta} \end{bmatrix} + E_1 f \tag{D21}$$

$$\begin{bmatrix} \dot{\hat{\phi}} \\ \dot{\hat{\eta}} \end{bmatrix} = (K_1A_1 + K_2A_3 - K_2^{-1}K_1^o(K_1A_1 + K_2A_4) \begin{bmatrix} x_1'' \\ \epsilon_1 \end{bmatrix}$$

$$+ \left[(K_1A_2 + K_2A_4)K_2^{-1} + gK_2B_2\right]\begin{bmatrix} \hat{\phi} \\ \hat{\eta} \end{bmatrix} + \left(K_1E_1 + K_2E_2\right) f \tag{D22}$$

where

$$A_1 = \begin{bmatrix} A_{11} & 0 \\ F_{11} & \hat{A}_{11}^o \end{bmatrix} \quad A_2 = \begin{bmatrix} A_{12} & 0 \\ F_{12} & \hat{A}_{12}^o \end{bmatrix} \tag{D23}$$

$$A_3 = \begin{bmatrix} A_{21} & 0 \\ F_{21} & \hat{A}_{21}^o \end{bmatrix} \quad A_4 = \begin{bmatrix} A_{22} & 0 \\ F_{22} & \hat{A}_{22}^o \end{bmatrix} \tag{D24}$$

$$B_2 = \begin{bmatrix} B_2 & 0 \\ 0 & \alpha^{-1}H_2 \end{bmatrix}, E_1 = \begin{bmatrix} E_1 \\ \hat{E}_1 \end{bmatrix}, \ E_2 = \begin{bmatrix} E_2 \\ \hat{E}_2 \end{bmatrix} \tag{D25}$$

Lemma C2 shows that

$$\begin{bmatrix} \hat{\phi}(t) \\ \hat{\eta}(t) \end{bmatrix} = e^{gK_2B_2t} \begin{bmatrix} \hat{\phi}(0) \\ \hat{\eta}(0) \end{bmatrix} + 0(g^{-1}) \tag{D26}$$

and

$$\begin{bmatrix} x_1''(t) \\ \varepsilon_1(t) \end{bmatrix} = e^{(A_1-A_2K_2^{-1}K_1)t} \begin{bmatrix} x_1''(0) \\ \varepsilon_1(0) \end{bmatrix} + \int_0^t e^{(A_1-A_2K_2^{-1}K_1)(t-\tau)} E_1 f(\tau)d\tau$$

$$+ 0(g^{-1}). \tag{D27}$$

providing that K_2B_2 is a stable matrix. From (D18)

$$\begin{bmatrix} \hat{\phi} \\ \hat{\eta} \end{bmatrix} = K_1 \begin{bmatrix} x_1'' \\ \varepsilon_1 \end{bmatrix} + K_2 \begin{bmatrix} x_2'' \\ \varepsilon_2 \end{bmatrix}. \tag{D28}$$

It can be recognized from (D26) that $[\hat{\phi} \quad \hat{\eta}]^T$ are the fast variables and from (D27) that $[x_1'' \quad \varepsilon_1]^T$ are the slow variables. Our procedure to decompose the fast and slow variables follows closely that in Section 2.1. The variables x and e can be expressed in terms the fast and slow variables as

$$\begin{bmatrix} x \\ e \end{bmatrix} = \Psi \begin{bmatrix} x_1'' \\ \varepsilon_1 \end{bmatrix} + \begin{bmatrix} B_o & 0 \\ 0 & \alpha^{-1}H \end{bmatrix} \left\{ \left(\begin{bmatrix} K & -K \\ \delta C & C_o \end{bmatrix} \begin{bmatrix} B_o & 0 \\ 0 & \alpha^{-1}H \end{bmatrix} \right)^{-1} \right\} \begin{bmatrix} \hat{\phi} \\ \hat{\eta} \end{bmatrix} \tag{D29}$$

where Ψ is a $2n \times (n-m-r)$ matrix whose columns span the null space of $\begin{bmatrix} K & -K \\ \delta C & C_o \end{bmatrix}$, that is,

$$\begin{bmatrix} K & -K \\ \delta C & C_o \end{bmatrix} \Psi = 0 \tag{D30}$$

VITA

Kar-Keung David Young was born in Hong Kong on December 3, 1952. He received the B.S. in Computer Engr. and M.S. in Electrical Engr. degrees from the University of Illinois, Urbana in 1974 and 1975, respectively. Since June 1974, he has been a research assistant in the Decision and Control Laboratory at the University of Illinois, Urbana, initially working on robot arm control algorithms and later on variable structure control systems.

Mr. Young is a member of Eta Kappa Nu.